UNIVERSITY OF CAMBRIDGE

MAMA/305, NST3AS/305, MAAS/305

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 2:00pm to 5:00pm

PAPER 305

THE STANDARD MODEL

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what it means for a global symmetry to be spontaneously broken.

A classical theory has a global continuous symmetry G which is spontaneously broken to H. State Goldstone's theorem in this context.

Let $M(x)$ be an $N \times N$ complex matrix field with action

$$S = \int d^4x \ \mathrm{Tr} \left(\partial^\mu M^\dagger \partial_\mu M - k\, M^\dagger M - \frac{\lambda}{2}\, M^\dagger M M^\dagger M \right)$$

with $\lambda > 0$.

(i) What is the global, continuous symmetry G of this theory? Show explicitly that all terms in the action are invariant under G.

(ii) Show that the symmetry is spontaneously broken if $k < 0$, with the ground state obeying

$$M_0^\dagger M_0 = v^2 \mathbb{1}$$

for some v^2. What is the unbroken symmetry group? How many Goldstone bosons does Goldstone's theorem imply?

(iii) By parameterising the ground state in terms of a suitable unitary matrix $U \in U(N)$, derive an appropriate action for the Goldstone bosons.

(iv) Suppose now that M is restricted to be a Hermitian matrix, i.e. $M^\dagger = M$. What is the global symmetry group of the theory? Show that the vacuum manifold has disconnected components and use Goldstone's theorem to determine the number of Goldstone bosons in each.

2 A Lie algebra-valued gauge potential $A_\mu = A_\mu^A T^A$ has field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

with g the gauge coupling.

(i) Under an infinitesimal gauge transformation, parameterised by the Lie algebra-valued $\alpha(x)$, the gauge potential transforms as $\delta A_\mu = \mathcal{D}_\mu \alpha = \partial_\mu \alpha - ig[A_\mu, \alpha]$. Determine the transformation of the field strength.

(ii) Determine the transformation of A_μ and $F_{\mu\nu}$ under charge conjugation C, parity P, and time-reversal T.

(iii) The Yang-Mills action, together with the theta term, is given by

$$S_{\text{YM}} = \int d^4x \, \left(-\frac{1}{2} \text{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{\theta g^2}{16\pi^2} \text{Tr}\, F_{\mu\nu} {}^\star F^{\mu\nu} \right)$$

where ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Show that both terms are invariant under infinitesimal gauge transformations. Determine how each term transforms under the discrete symmetries C, P and T.

(iv) Show that the theta term can be written as $\text{Tr}\, F_{\mu\nu} {}^\star F^{\mu\nu} = \partial_\mu K^\mu$ with

$$K^\mu = 2\epsilon^{\mu\nu\rho\sigma} \text{Tr} \left(A_\nu \partial_\rho A_\sigma - \frac{2ig}{3} A_\nu A_\rho A_\sigma \right)$$

Explain why the theta term does not contribute to the classical equations of motion.

(v) The Yang-Mills field is coupled to N_f massless Dirac fermions, each in the fundamental representation, with total action

$$S = S_{\text{YM}} - i \int d^4x \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu \mathcal{D}_\mu \psi_i$$

Under an axial rotation, $\psi_i \to e^{i\beta\gamma^5}\psi_i$, the chiral anomaly means that the theta parameter transforms as $\theta \to \theta + 2N_f\beta$. Use Noether's theorem to derive the result

$$\partial_\mu J_A^\mu = \alpha \text{Tr} F_{\mu\nu} {}^\star F^{\mu\nu}$$

where J_A^μ is the axial current and α is a constant, both of which you should find as part of the derivation.

(vi) Show that it's possible to define a new axial current $\tilde{J}_A^\mu$ that is conserved, with $\partial_\mu \tilde{J}_A^\mu = 0$, but is not gauge invariant.

3 Briefly describe the implications if a symmetry has:

- a gauge anomaly,
- a chiral, or ABJ, anomaly,
- a 't Hooft anomaly.

An $SU(N)$ gauge theory is coupled to a single left-handed Weyl fermion λ in the anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation and p left-handed Weyl fermions ψ^i, $i = 1, \ldots, p$, each in the anti-fundamental $\overline{\square}$ representation.

(i) For what value of p is the quantum theory consistent?

(ii) Write down the classical, global symmetries of the theory. Show that the quantum theory has a $SU(p) \times U(1)$ global symmetry.

(iii) Compute the $SU(p)^3$, $SU(p)^2 U(1)$, $U(1)^3$ and mixed $U(1)$-gravitational 't Hooft anomalies for this theory.

(iv) It is conjectured that this theory confines without spontaneously breaking any global symmetry. The massless degrees of freedom are given by the left-handed gauge singlet

$$\chi^{ij} = \psi^{(i}(\lambda\psi^{j)}) \quad i, j = 1, \ldots, p$$

transforming in the symmetric $\square\square$ representation of the $SU(p)$ global symmetry. Show that the 't Hooft anomalies of the fermion χ match those of the original gauge theory.

[*You may use the fact that the dimension, Dynkin index I and anomaly coefficient A of various $SU(N)$ representations are given by:*

R	$\square$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\dim(R)$	N	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$N+2$	$N-2$
$A(R)$	1	$N+4$	$N-4$

.]

4 Denote the $U(1)$ hypercharge gauge field as B_μ and the $SU(2)$ weak gauge field as $W_\mu = \frac{1}{2} W_\mu^A \sigma^A$, with σ^A the usual Pauli matrices. Consider the action

$$S = \int d^4x \left[-\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} \mathrm{Tr}\, W_{\mu\nu} W^{\mu\nu} + \mathcal{D}_\mu H^\dagger \, \mathcal{D}^\mu H - \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2 \right]$$

where H is the Higgs field and $\mathcal{D}_\mu H = \partial_\mu H - igW_\mu H - \frac{i}{2} g' B_\mu H$.

(i) By parameterising the general form of the Higgs field as

$$H = e^{i\xi^A(x)\sigma^A} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

derive an action for the propagating fields in the theory and determine their masses in terms of the couplings g, g' and λ and the Higgs expectation value v.

(ii) Find the generator of the unbroken $U(1)$ gauge symmetry. Use this to determine the charges of each of the massive bosonic excitations.

(iii) List the fundamental fermions in one generation of the Standard Model, including the right-handed neutrino, and give their representations under the gauge group.

(iv) Explain why it's not possible to write down Dirac mass terms for the fermions but how it is possible for them all to gain a mass through Yukawa interactions with the Higgs. Use the expression for the unbroken generator derived in (iii) to determine the charges of each of these fermions under the unbroken $U(1)$ symmetry.

END OF PAPER