UNIVERSITY OF CAMBRIDGE

MAMA/304, NST3AS/304, MAAS/304, MAPY/2/AQFT, NST3PHY/2/AQFT

MAT3

MATHEMATICAL TRIPOS Part III

Thuesday 11 June 2026 2:00pm to 5:00pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A scalar field theory in four dimensional Minkowski space has Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{\delta_Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4!}\phi^4.$$

(a) Briefly explain the significance of each term in the Lagrangian.

(b) (i) Draw all 1PI diagrams that contribute to the quartic term in the quantum effective action $\Gamma_4(s,t,u)$ up to and including order λ^2. Here s, t and u are the Mandelstam variables.

(ii) Evaluate these diagrams by introducing a large momentum cut-off Λ, expressing your answers as an integral over a parameter x. By requiring that

$$\Gamma_4(M^2, M^2, M^2) = \lambda,$$

(i.e. where $s = t = u = M^2$), fix the counter-term δ_λ and show that if $m \ll M \ll \Lambda$, then $\Gamma_4(s,t,u)$ may be written as

$$\Gamma_4(s,t,u) = \lambda - \frac{\lambda^2}{32\pi^2}\int_0^1 dx\, \ln\left(\frac{F^3(M^2)}{F(s)F(t)F(u)}\right),$$

to one-loop, where the function $F(p^2)$ is defined as $F(p^2) = m^2 - x(1-x)p^2$.

(c) Show that, in order for $\Gamma_4(s,t,u)$ to be independent of M, λ must be a function of M. Hence find λ as an explicit function of M to leading order.

[*Hint: You may use the following integral*

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2},$$

without proof.]

2

(a) In d-dimensional Minkowski spacetime, the Lagrangian of QED may be written as

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}A\!\!\!/\psi - \frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\delta_2\partial\!\!\!/ - \delta_m)\psi - e\delta_1\bar{\psi}A\!\!\!/\psi,$$

using standard notation.

(i) Draw all Feynman diagrams that are needed to calculate the leading order contributions to the counter-terms δ_2, δ_m and δ_1.

(ii) Using dimensional regularization in $d = 4 - \epsilon$ dimensions, in the MS scheme, show that

$$\delta_2 = -\frac{e^2}{8\pi^2\epsilon}, \qquad \delta_m = -\frac{me^2}{2\pi^2\epsilon},$$

to leading order.

(b) Now consider a scalar field theory in three-dimensional Minkowski spacetime with Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{6!}\phi^6 + \frac{\delta_Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{6!}\phi^6.$$

(i) Draw all diagrams that you would need to evaluate to calculate the leading order contributions to the counter-terms δ_m and δ_λ (you do not have to evaluate the diagrams).

(ii) By considering the superficial degree of divergence or otherwise, explain why

$$\delta_m \sim \frac{\Lambda^2}{m^2}, \qquad \text{and} \qquad \delta_\lambda \sim \ln(\Lambda/m),$$

where Λ is a UV cut-off.

(iii) Explain why there is no contribution to δ_Z at one and two loops.

[*Hint: You may use the following integrals without proof*

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}, \qquad \int_{\mathbb{R}^d} \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}.$$

The behaviour of the Gamma function near zero is

$$\Gamma(z) = \frac{1}{z} - \gamma + ...,$$

where γ is the Euler-Mascheroni constant. You may also use the d-dimensional gamma matrix identities $\gamma^\mu\gamma_\mu = d$ and $\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu$ without proof.]

3

(a) This part of the question concerns a theory in zero dimensions with two interacting scalar variables, ϕ and χ, with action

$$S(\phi,\chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}M^2\chi^2 + \lambda\phi^2\chi,$$

where m, M and λ are constants.

One can integrate out the χ variable to obtain an effective action $W(\phi)$ for the ϕ variable, given by

$$e^{-W(\phi)} = N\int_{-\infty}^{\infty} d\chi\, e^{-S(\phi,\chi)},$$

where N is a normalization constant.

(i) For a suitable choice of N, show that $W(\phi)$ can be written in the form

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \widetilde{\lambda}\phi^4,$$

where you should give an expression for $\widetilde{\lambda}$ in terms of λ and M.

(ii) By considering tree-level Feynman diagrams for $S(\phi,\chi)$, briefly comment on the relationship between $\widetilde{\lambda}$, λ and M.

(b) In d-dimensional Euclidean space, the action for two scalar fields $\phi(x)$ and $\chi(x)$ is given by

$$S[\phi,\chi] = S_1[\phi] + S_2[\chi] + S_3[\phi,\chi],$$

where

$$S_2[\chi] = \int d^dx\, \left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}M^2\chi^2\right).$$

(i) With a suitable choice of normalization, show that the effective action for the $\phi(x)$ fields, given by integrating out the $\chi(x)$ fields, is given by

$$W[\phi] = S_1[\phi] - \ln\left\{\exp\left(-S_3\left[\phi, -\frac{\delta}{\delta J}\right]\right) Z_0[J]\right\}_{J=0},$$

where $Z_0[J]$ should be given explicitly.

(ii) Let

$$S_3[\phi,\chi] = \int d^dx\, \lambda\phi^2(x)\chi(x).$$

Find an expression for the leading order correction to $W[\phi]$ in the effective action. Comment on how your result compares with that of part (a) in the limit in which M is much larger than any momentum.

4 A non-abelian gauge field $A_\mu(x)$ has generating functional

$$Z[J] = N \int \mathcal{D}A\, e^{iS[A] + i\int d^4x J^\mu(x) A_\mu(x)},$$

where N is an unspecified normalization constant. The gauge field transforms under infinitesimal gauge transformation with parameter $\alpha(x)$ as

$$A_\mu(x) \to A_\mu^\alpha(x) = A_\mu(x) + D_\mu \alpha(x),$$

where D_μ is a covariant derivative. You may assume that both the action $S[A]$ and the measure $\mathcal{D}A$ are gauge-invariant.

(a) The Faddeev-Popov determinant Δ_{FP} for a vector field $A_\mu(x)$ and gauge choice $G(A) = 0$ may be written as

$$1 = \Delta_{FP} \int \mathcal{D}\alpha\, \delta[G(A^\alpha)].$$

Show that

$$\Delta_{FP} = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)_{G=0}.$$

(b) Axial gauge is given by

$$G(A) = n^\mu A_\mu - \omega(x),$$

where n^μ is a fixed constant vector. $\omega(x)$ is an arbitrary function that may be introduced into the path integral with gaussian-like weighting so that

$$\frac{1}{N_\xi} \int \mathcal{D}\omega\, \exp\left(-\frac{i}{2\xi} \int d^4x\, \omega^2(x)\right) = 1,$$

for some suitably chosen constant N_ξ. Show that the generating functional for a gauge field $A_\mu(x)$ may be written in axial gauge as

$$Z[J] = N \int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c\, e^{iS[A] + iS_{\text{gh}}[c,\bar{c},A] + iS_{\text{gf}}[A] + i\int J^\mu A_\mu},$$

where $S_{\text{gh}}[c, \bar{c}, A]$ and $S_{\text{gf}}[A]$ are functionals you should derive. You may assume that Δ_{FP} is gauge-invariant.

(c) By considering the BRST transformation of the gauge-fixing fermion

$$\Psi_n(x) = \bar{c}_a(x) \left(n^\mu A_\mu^a(x) - \frac{\xi}{2} B^a(x)\right),$$

where a is a Lie algebra index, explain why the gauge-fixed action is invariant under BRST transformations. In the above expression $B^a(x)$ is related to the ghost field $\bar{c}_a(x)$ by the BRST transformation

$$\delta_Q \bar{c}^a(x) = \xi B^a(x).$$

You may assume $\delta_Q^2 = 0$ without proof.

(d) We now choose to change the axial gauge from the fixed vector n^μ to a different fixed vector u^μ. Explain why the physics is not changed by replacing n^μ with u^μ.

[*Hint: You may use without proof*

$$\det M = \int \mathcal{D}\bar{c}\mathcal{D}c \, \exp\left(i \int d^4x \, \bar{c}^a M_{ab} c^b \right) ,$$

for scalar Grassmann fields $\bar{c}(x)$ *and* $c(x)$.]

END OF PAPER