UNIVERSITY OF CAMBRIDGE

MAMA/303, NST3AS/303, MAAS/303

MAT3

MATHEMATICAL TRIPOS Part III

Wednesay 10 June 2026 9:00am to 11:00am

PAPER 303

STATISTICAL FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a mean field theory with order parameter m and Z_2-symmetric effective free energy

$$F = \frac{\alpha_2(T)}{2} m^2 + \frac{\alpha_4(T)}{4} m^4 + \ldots$$

(i) Assume that $\alpha_2(T) > 0$ for $T > T_c$, $\alpha_2(T)$ has a simple zero at $T = T_c$ and $\alpha_4(T) > 0$. Explain briefly how this model describes a second order phase transition at $T = T_c$ and compute the critical exponents α, β.

[Recall $c \sim |T - T_c|^{-\alpha}$, $m \sim (T_c - T)^{\beta}$.]

(ii) Now consider a *tricritical point* where α_2 and α_4 are both positive for $T > T_c$, both have a simple zero at $T = T_c$ and the next term in F is $\frac{\alpha_6(T)}{6} m^6$ where $\alpha_6(T) > 0$. Determine the critical exponents α, β at this point.

(b) Consider a Landau-Ginzburg effective free energy containing only quadratic terms:

$$F = \frac{1}{2} \int d^d x \left(\mu^2 \phi^2 + \gamma (\nabla \phi)^2 + \sum_{n \geqslant 2} \delta_n \phi (-\nabla^2)^n \phi \right)$$

where the parameters μ^2, γ and δ_n depend analytically on T, $\mu^2 > 0$ for $T > T_c$, μ^2 has a simple zero at $T = T_c$ and $\gamma > 0$.

(i) Explain carefully how to calculate the partition function and hence show that fluctuations in ϕ contribute a term to the thermodynamic free energy per unit volume F_{thermo}/V of the form

$$-\frac{T}{2} \int \frac{d^d k}{(2\pi)^d} \log \left(\frac{\pi V T}{K(T, \mathbf{k}^2)} \right)$$

with $K(T, \mathbf{k}^2) = \mu^2(T) + \gamma(T) \mathbf{k}^2 + (\mathbf{k}^2)^2 g(T, \mathbf{k}^2)$ for some function $g(T, \mathbf{k}^2)$ to be determined and the integral is over all $\mathbf{k}$ with $|\mathbf{k}| < \Lambda$.

Henceforth you may assume $g(T, \mathbf{k}^2) > 0$ for all $\mathbf{k}$ and you need only consider $T > T_c$.

(ii) Show that fluctuations in ϕ make a contribution to the heat capacity per unit volume c that includes a term of the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{A(T, \mathbf{k}^2)}{K(T, \mathbf{k}^2)^2}$$

where you should give an expression for A in terms of K and show that A is non-vanishing at $T = T_c$, $\mathbf{k} = 0$.

(iii) For the critical point of (a)(i) use the above result to determine the contribution of quadratic fluctuations to the critical exponent α and hence determine the upper critical dimension.

(iv) Do the same for the tricritical point of (a)(ii).

2

(a) Explain briefly the three steps of the renormalization group procedure for a Landau-Ginzburg theory defined by an effective free energy $F[\phi]$.

(b) Consider a theory with effective free energy $F = \int d^dx \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + g\phi^4\right)$ where $g \geqslant 0$. Ignoring contributions from couplings other than μ^2 and g, the β functions are

$$\beta_{\mu^2} = 2\mu^2 + \frac{12\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^d g}{\Lambda^2+\mu^2} + O(g^2)$$

$$\beta_g = (4-d)g - \frac{36\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^d g^2}{(\Lambda^2+\mu^2)^2} + O(g^3)$$

where Ω_n is the area of a n-dimensional unit sphere and Λ is the UV cut-off.

(i) Determine the scaling dimensions of operators at the gaussian fixed point, stating which linear combinations of operators have these scaling dimensions.

(ii) Explain why the critical surface of the gaussian fixed point does not describe the Ising universality class if $d < 4$.

(iii) Let $d = 4 - \epsilon$ with $\epsilon > 0$ and define $\tilde{g} = \Lambda^{d-4}g$. Assume ϵ is small. Show that there is a Wilson-Fisher fixed point at

$$\mu_\star^2 = -\frac{\Lambda^2}{6}\epsilon + O(\epsilon^2) \qquad \tilde{g}_\star = \frac{2\pi^2}{9}\epsilon + O(\epsilon^2)$$

[You may use $\Omega_3 = 2\pi^2$.]

(iv) Determine the scaling dimensions of couplings at the Wilson-Fisher fixed point, stating which linear combinations of couplings these correspond to. State whether they are relevant, irrelevant or marginal.

(c) Now include a term $h\phi^6$ in F. Assume that initially $\mu^2 = \mu_0^2$, $g = g_0$ and $h = h_0$ where g_0 and h_0 are small. Step 1 of the renormalization group procedure gives

$$\mu'^2 = \mu_0^2 + 12g_0I_1 + Ah_0I_1^2 + O(g_0^2) + O(g_0h_0) + O(h_0^2)$$

$$g' = g_0 + Bh_0I_1 - 36g_0^2I_2 + O(g_0^3) + O(g_0h_0) + O(h_0^2)$$

$$h' = h_0 + Cg_0h_0I_2 + O(h_0^2) + O(g_0^2h_0) + O(g_0^3)$$

where A, B, C are constants and

$$I_n = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^dq}{(2\pi)^d(q^2+\mu_0^2)^n}$$

(i) Draw the Feynman diagrams that contribute to A, B, C. [You don't need to show diagrams related by symmetries.]

(ii) Calculate A, B, C.

(iii) By considering the β function for h show that $h = O(g^3)$ at a fixed point. Deduce that the result of (b)(iii) is unaffected by the inclusion of the ϕ^6 term. [Lengthy calculations are not required!]

3

(a) In Landau-Ginzburg theory, consider the fixed point of the renormalization group flow that describes the Ising universality class.

(i) Show that under a renormalization group transformation the rescaled field satisfies $\phi'(\mathbf{x}') = \zeta^{\Delta_\phi}\phi(\mathbf{x})$ for some Δ_ϕ.

(ii) Show that $\Delta_t = 1/\nu$ where Δ_t is the scaling dimension of the reduced temperature $t = (T - T_c)/T_c$ and ν is defined by $\xi \sim t^{-\nu}$.

(iii) Show that the magnetic field B has scaling dimension $\Delta_B = (d + 2 - \eta)/2$. [You may assume $\Delta_\phi = (d - 2 + \eta)/2$.]

(iv) Use the definitions $c \sim t^{-\alpha}$, $\langle\phi\rangle \sim t^{\beta}$, $\chi \sim t^{-\gamma}$, $\langle\phi\rangle \sim B^{1/\delta}$ to derive the scaling relations

$$\alpha = 2 - d\nu \qquad \beta = \frac{1}{2}(d - 2 + \eta)\nu \qquad \gamma = \nu(2 - \eta) \qquad \delta = \frac{d + 2 - \eta}{d - 2 + \eta}$$

(v) For $d > 4$ the correct value of δ is 3, which disagrees with the above expression. Explain the reason for this disagreement and show that $\delta = 3$ is consistent with scaling. Why does this disagreement not occur for $d < 4$?

(b) A system in d spatial dimensions is described by two complex order parameters ψ_1 and ψ_2. The effective free energy is

$$F = \int d^d x \left[\gamma(\nabla\psi_1^\star \cdot \nabla\psi_1 + \nabla\psi_2^\star \cdot \nabla\psi_2) + \mu_1^2|\psi_1|^2 + \mu_2^2|\psi_2|^2 + g(|\psi_1|^2 + |\psi_2|^2)^2\right]$$

where $\gamma > 0$ and $g > 0$. Use mean field theory to predict the phase diagram in the (μ_1^2, μ_2^2) plane, paying attention to any special lines in this diagram. What are the unbroken symmetries in each phase? How many Goldstone bosons are present in each phase?

END OF PAPER