MAMA/302, NST3AS/302, MAAS/302

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 9:00am to 12:00pm

PAPER 302

SYMMETRIES, PARTICLES, AND FIELDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Lie group $G = SO(1,1)$ is the set of 2×2 real matrices such that

$$SO(1,1) = \left\{ \Lambda \in GL(2,\mathbb{R}) \;\middle|\; \Lambda^{\mathsf{T}} \eta \Lambda = \eta \,, \det \Lambda = 1 \right\} \quad \text{where } \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(a) Show that any element of the Lie algebra $X \in \mathfrak{g} = \mathfrak{so}(1,1)$ must satisfy

$$X^{\mathsf{T}} \eta + \eta X = 0$$

and give a basis for the algebra, showing that it is one-dimensional.

(b) Let $X(\phi) \in \mathfrak{so}(1,1)$, where ϕ is a real parameter. Using the exponential map, determine the matrix elements of $\Lambda(\phi) := \exp X(\phi)$ in terms of $\sinh\phi$ and $\cosh\phi$. Show that the set of elements $\Lambda(\phi)$ forms a subgroup of $SO(1,1)$. Explain whether this subgroup is Abelian and whether it is compact, justifying your answers.

(c) The subgroup in part (b) acts on vectors $(x,y)^{\mathsf{T}} \in \mathbb{R}^2$ by matrix multiplication. Show how this subgroup acts on coordinates $u = x+y$ and $v = x-y$. Use this to interpret the subgroup transformations geometrically. What curves in the x-y plane are invariant under the action of this subgroup?

(d) Determine whether $SO(1,1)$ is connected, justifying your answer. [*Hint: It may be useful to consider the matrix*

$$K = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} .]$$

(e) Let the *Cayley transform*, $C : \mathfrak{g} \to G$, be defined to be

$$C(X) := (I + X)(I - X)^{-1} .$$

For $X(\phi) \in \mathfrak{so}(1,1)$, determine the 2×2 matrix given by $C(X(\phi))$ in terms of ϕ. Demonstrate that $C(X(\phi)) \in SO(1,1)$. Does the image of the Cayley transform differ from the image of the exponential map? If so, how?

2 Consider the Lorentz group

$$O(1,3) = \left\{ \Lambda \in GL(4,\mathbb{R}) \,\middle|\, \Lambda^{\mathsf{T}} \eta \Lambda = \eta \right\}$$

with Minkowski metric $[\eta^{\mu\nu}] = \mathrm{diag}(1,-1,-1,-1)$.

(a) Show that the generators of the Lorentz algebra can be represented by matrices $M^{\mu\nu}$, with matrix elements given by

$$(M^{\mu\nu})^{\alpha}{}_{\beta} = \eta^{\mu\alpha}\delta^{\nu}{}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}{}_{\beta}\,.$$

Explain, with justification, which of these, denoted J_i, is a generator of spatial rotations about the x_i axis ($i = 1,2,3$), and which of these, denoted K_i generate Lorentz boosts in the x_i direction.

(b) By deriving results for the Lie brackets $[J_1, J_2]$, $[K_1, K_2]$, and $[J_1, K_2]$, or otherwise, deduce expressions for the full set of Lie brackets, and show that the complexified Lorentz algebra is isomorphic to $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. Briefly explain how this allows us to categorize irreducible representations of the Lorentz group. [*You do not need to discuss taking real forms in detail.*]

(c) The parity operator $[P^{\alpha}{}_{\beta}] = \mathrm{diag}(1,-1,-1,-1)$ acts on coordinates x^{ν} as $(x^0, \mathbf{x}) \mapsto (x^0, -\mathbf{x})$. Explain why this implies that $M^{\mu\nu} \mapsto PM^{\mu\nu}P$ under parity. In terms of the categorization in part (b), which irreducible representations of the Lorentz algebra are invariant under parity? What does this imply for Weyl fermions and Dirac fermions?

(d) Let the dual generators be defined as $\widetilde{M}_{\rho\sigma} := \frac{1}{2}\epsilon_{\rho\sigma\mu\nu}M^{\mu\nu}$, where $\epsilon_{0123} = 1$ and is antisymmetric under permutation of its indices. Further define

$$\begin{aligned} C_1 &= M_{\mu\nu}M^{\mu\nu} \\ C_2 &= M_{\mu\nu}\widetilde{M}^{\mu\nu}\,. \end{aligned}$$

Determine how C_1 and C_2 transform under parity. Comment on the relationship between these two objects and the isomorphism found in part (b).

3 Let $\mathfrak{g}$ be a Lie algebra.

(a) Give a definition for the adjoint representation, ad, of $\mathfrak{g}$ and show that it satisfies the required properties of a representation.

(b) Give a definition for the Killing form κ of $\mathfrak{g}$ and prove that

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])$$

for all X, Y, $Z \in \mathfrak{g}$.

(c) Denote the structure constants of $\mathfrak{g}$ by $f^d{}_{ab}$, or equivalently, by $f_{ab}{}^d$. Let $f_{abc} := f^d{}_{ab}\kappa_{dc}$, where $\kappa_{dc} := \kappa(T_d, T_c)$ and $\{T_a\}$ are basis elements of $\mathfrak{g}$. Show that f_{abc} is totally antisymmetric in a, b, and c.

(d) Suppose that $\mathfrak{g}$ is simple and of compact type. Prove that the adjoint representation is irreducible. [*Hint: Consider a proof by contradiction.*]

(e) Let d be an antihermitian, irreducible representation of $\mathfrak{g}$ acting on vector space V, with $\dim V > 1$. Let

$$H(X,Y) := \mathrm{Tr}(d(X)\,d(Y))$$

for all X, $Y \in \mathfrak{g}$. Show that

$$H([X,Y],Z) = H(X,[Y,Z])$$

for all X, Y, $Z \in \mathfrak{g}$. Further show that, in an adapted basis for $\mathfrak{g}$,

$$H(T_a, T_b) = -\mu\,\delta_{ab}\,,$$

where μ is a positive constant. [*You may use Schur's lemma without proof.*]

(f) Let

$$B(X,Y,Z) := \mathrm{Tr}(d(X)\,d(Y)\,d(Z))$$

for all X, Y, $Z \in \mathfrak{g}$. By considering $B([X,Y],Z,W)$ prove

$$f^e{}_{ba}B_{cde} + f^e{}_{bc}B_{dae} + f^e{}_{bd}B_{ace} = 0$$

where $B_{abc} = B(T_a, T_b, T_c)$. Further show that B satisfies the identity

$$f_a{}^{bc}B_{bcd} = -\frac{\mu}{2}\,\delta_{ad}$$

where indices are raised with the inverse of the Killing form.

4 This question concerns the matrix Lie group $SO(5)$.

(a) State the definition of $SO(5)$ and find its dimension, denoted D.

(b) Show explicitly that $SO(5)$ has an $SO(4)$ subgroup.

(c) Consider the following two representations of $SO(5)$

(i) the 5-dimensional representation, that is, the vector space acted on by the $SO(5)$ matrices,

(ii) the D-dimensional, adjoint representation.

When restricted to the $SO(4)$ subgroup from part (b), show how vectors in these representations reduce to direct sums of $SO(4)$ irreducible representations. Be sure to check the dimensions of the $SO(4)$ representations and their actions under group transformations. You *do not* need to check irreducibility. [*Hint: Use your answer to part (b).*]

(d) Let us assume a basis for the $\mathfrak{so}(5)$ root space (denoted $\mathfrak{h}^*_{\mathbb{R}}$ in the lectures) such that the simple roots $\alpha_{(1)}$ and $\alpha_{(2)}$ have coordinates $(1, 0)$ and $(-1, 1)$, respectively. Determine the coordinates of the fundamental weights $\omega_{(1)}$ and $\omega_{(2)}$. Sketch the weight lattice, including the fundamental weights and the full root set.

(e) Find the full weight sets for the representations, denoted $d_{[1,0]}$ and $d_{[0,1]}$, whose highest weights are respectively $\omega_{(1)}$ and $\omega_{(2)}$. State the dimensions of these representations. Specify the highest weight of the adjoint representation, justifying your answer. [*You may assume that all (nonzero) roots have multiplicity 1.*]

(f) Once again restricting from $SO(5)$ to the $SO(4)$ subgroup of part (b), discuss how the representations found in part (e) reduce to the algebra representations which can be inferred from parts (b) and (c), or to representations of other Lie algebras related by isomorphism.

END OF PAPER