MAT3

MATHEMATICAL TRIPOS Part III

Thursday 4 June 2026 9:00am to 12:00pm

PAPER 301

QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a four-dimensional Lagrangian for a free Dirac spinor field,

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \ .$$

Here m is the mass of the field, and γ^μ are matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

(a) Under a rigid translation, $x^\mu \to x^\mu - \epsilon^\mu$, the spinor field transforms as

$$\delta\psi(x) = \epsilon^\mu \partial_\mu \psi(x) \ .$$

For the Lagrangian $\mathcal{L}_0$, construct the Noether current associated to this transformation. Express the current as $j^\mu = \epsilon_\nu T^{\mu\nu}$, and show that $\partial_\mu T^{\mu\nu} = 0$ on-shell.

(b) Using only combinations of the form

$$\bar{\psi}\gamma^\nu \partial^\mu \psi \ , \quad (\partial^\mu \bar{\psi})\gamma^\nu \psi \ , \quad \bar{\psi}\gamma^\nu \gamma^\mu \psi \ , \quad \bar{\psi}\gamma^\nu \gamma^\mu \gamma^\sigma \partial_\sigma \psi \ , \quad (\partial_\sigma \bar{\psi})\gamma^\nu \gamma^\mu \gamma^\sigma \psi \ ,$$

plus similar terms with all possible permutations of the gamma matrices, construct the most general 2-tensor $\hat{T}^{\mu\nu}$ which satisfies,

$$\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu} \ ,$$

$$\partial_\mu \hat{T}^{\mu\nu} = 0 \ ,$$

when the equation of motion hold. In other words, $\hat{T}^{\mu\nu}$ is a symmetric and conserved tensor on-shell. You are allowed to simplify your expression for $\hat{T}^{\mu\nu}$ by using equations of motion.

(c) By choosing an appropriate overall coefficient for the 2-tensor $\hat{T}^{\mu\nu}$, show that

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \frac{i}{4}\left(\partial_\lambda S^{\lambda\mu\nu} - \partial^\nu(\bar{\psi}\,\gamma^\mu\,\psi)\right) \ ,$$

where

$$S^{\lambda\mu\nu} = \frac{1}{2}\bar{\psi}\,[\gamma^\lambda, \gamma^\mu]\gamma^\nu\,\psi \ .$$

Construct the conserved charges associated to $\hat{j}^\mu = \epsilon_\nu \hat{T}^{\mu\nu}$ and $j^\mu = \epsilon_\nu T^{\mu\nu}$. Show explicitly that the resulting charges are equal up to boundary terms.

2 In this question we will be working with a free real scalar field $\phi(x)$ of mass m, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \ .$$

Upon quantizing the field, the mode decomposition is given by

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x}\right) \ ,$$

where $\omega = \sqrt{\vec{k}^2 + m^2}$ and

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') \ , \qquad [a_{\vec{k}}, a_{\vec{k}'}] = 0 \ .$$

A parity transformation is a spatial inversion that takes $\vec{x} \to -\vec{x}$. When acting on the scalar field it is defined by

$$\phi(\vec{x}, t) \ \to \ \mathcal{P}\phi(\vec{x}, t)\mathcal{P}^{-1} = \gamma_p\, \phi(-\vec{x}, t) \ , \qquad (\star)$$

where the parity operator $\mathcal{P}$ is a unitary operator which leaves the vacuum invariant, $\mathcal{P}|0\rangle = |0\rangle$, and $\gamma_p = \pm 1$ is called the intrinsic parity of the field.

(a) Show that parity leaves invariant the action of the free theory with Lagrangian $\mathcal{L}$.

(b) Show that for an arbitrary n-particle state

$$\mathcal{P}|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle = (\gamma_p)^n| - \vec{k}_1, -\vec{k}_2, \ldots, -\vec{k}_n\rangle \ .$$

(c) Prove the following relations

$$\mathcal{P}_1\, a_{\vec{k}}\, \mathcal{P}_1^{-1} = i a_{\vec{k}} \ , \qquad \mathcal{P}_2\, a_{\vec{k}}\, \mathcal{P}_2^{-1} = -i\gamma_p\, a_{-\vec{k}} \ ,$$

where

$$\mathcal{P}_1 = \exp\Big(-i\frac{\pi}{2}\int \frac{d^3k}{(2\pi)^3} a^\dagger_{\vec{k}}\, a_{\vec{k}}\Big) \ , \qquad \mathcal{P}_2 = \exp\Big(i\frac{\pi}{2}\gamma_p \int \frac{d^3k}{(2\pi)^3} a^\dagger_{\vec{k}}\, a_{-\vec{k}}\Big) \ .$$

[*Hint: You may use without proof that, for any operators A and B,*

$$e^{i\beta A}Be^{-i\beta A} = \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}B_n$$

where $B_0 = B$ and $B_n = [A, B_{n-1}]$, for $n = 1, 2, \ldots$.]

(d) Show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ are unitary operators. Show that the operator $\mathcal{P}_1\mathcal{P}_2$ implements $(\star)$, that is, $\mathcal{P}_1\mathcal{P}_2$ gives an explicit expression for the parity operator $\mathcal{P}$ acting on the field ϕ.

3 Suppose we have a gauge field A_μ in four dimensions with a Lagrangian given by

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \ ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and α is a constant. The Feynman propagator is given by

$$\Delta_{\mu\nu}(x-y) = \langle 0|\mathrm{T}A_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i\Pi_{\mu\nu}}{k^2 - i\epsilon}e^{ik\cdot(x-y)}$$

with

$$\Pi_{\mu\nu} = -\left(\eta_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2}\right) \ .$$

(a) Verify that $\Delta_{\mu\nu}(x-y)$ is the Green's function for an appropriate second order differential equation that you should specify.

(b) Consider adding an external non-dynamical current j^μ that couples to the gauge field A_μ, so that the Lagrangian is modified to

$$\mathcal{L} = \mathcal{L}_0 - e\, j^\mu A_\mu \ , \qquad (\star)$$

where $\partial_\mu j^\mu = 0$ and e is a coupling constant. For this theory, described by the Lagrangian ($\star$), state the Schwinger-Dyson equation for the two-point function $\langle\Omega|\mathrm{T}A_\mu(x)A_\nu(y)|\Omega\rangle$, where $|\Omega\rangle$ is the vacuum of the interacting theory. Briefly state the appropriate assumptions and their relevance in the Schwinger-Dyson equation.

(c) Using the Schwinger-Dyson equation, write $\langle\Omega|\mathrm{T}A_\mu(x)A_\nu(y)|\Omega\rangle$ in terms of appropriate integrals of $\Delta_{\mu\nu}(x-y)$ and j^μ to order e^2. Express your answer in terms of Feynman diagrams, and briefly comment on why the expression truncates in e.

(d) Consider this approach applied to QED, where the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is dynamical, and $\mathcal{L}_0$ is now the Lagrangian the gauge field and free massive Dirac field ψ. Do you expect the same conclusion as in part (c)? Briefly explain your answer and draw the connected Feynman diagrams one would have to compute to evaluate the two-point function to order e^4 (you do not need to evaluate the diagrams).

4 Consider scalar QED, a four-dimensional theory consisting of a complex scalar field φ of mass m and an electromagnetic gauge field A_μ, with the only interaction being due to minimal coupling between them, with coupling constant e. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathcal{D}_\mu\varphi)^*(\mathcal{D}^\mu\varphi) - m^2\varphi\varphi^* \ ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\mathcal{D}_\mu\varphi = \partial_\mu\varphi + ieA_\mu\varphi \ .$$

(a) Identify the free and interaction terms in the Lagrangian. In the interaction Lagragian, is the coupling marginal, relevant or irrelevant? Explain briefly your answer.

(b) Write, without derivation, the corresponding Feynman rules in momentum space for the S-matrix.

(c) Consider the scattering process $\varphi\varphi^* \to \gamma\gamma$, where two scalars of opposite charge transition to two photons. Draw the connected Feynman diagrams that contribute to the S-matrix to leading order in e.

(d) Based on your answers to part (c) and (b), evaluate the connected contribution to $\varphi\varphi^* \to \gamma\gamma$ scattering at leading order in e. Verify that the appropriate Ward identity, where a polarisation vector is replaced by momentum, holds for this process.

END OF PAPER