MAMA/226, NST3AS/226, MAAS/226

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday 16 June 2026 9:00am to 11:00am

PAPER 226

RANDOM WALKS AND PHASE TRANSITIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\Gamma = (G, \mu, \kappa)$ denote a locally finite weighted graph with symmetric weights μ and killing measure κ. Let $X = (X_n)_{n \geqslant 0}$ denote the associated discrete-time random walk, with canonical law P_x when started from $x \in G$. For $A \subset G$, we define

$$\begin{aligned} H_A &= \inf\{n \geqslant 0 : X_n \in A\}, \\ \widetilde{H}_A &= \inf\{n \geqslant 1 : X_n \in A\}, \\ L_A &= \sup\{n \geqslant 0 : X_n \in A\}, \end{aligned}$$

with the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$.

(a) Assume that $G = \mathbb{Z}^d$, $d \geqslant 3$ with natural weights (i.e. $\mu_{x,y} = 1_{\{|x-y|=1\}}$ and $\kappa_x = 0$, for all $x, y \in \mathbb{Z}^d$). Let $x_k = 2^k e$ where $e = (1, 0, \ldots, 0) \in \mathbb{Z}^d$ and consider

$$A = \{x_k : k = 0, 1, 2, \ldots\}.$$

Using a Borel-Cantelli argument, show that

$$P_0(X_n \in A \text{ for infinitely many } n) = 0.$$

Hint: consider the events $\{H_{\{x_k\}} < \infty\}$.

(b) Assume that Γ is transient.

(i) Let A, B be finite subsets of G. Show that

$$\mu_y P_y(L_B \leqslant \widetilde{H}_A, L_B \geqslant 0) = P_{e_B}(X_{H_A} = y, H_A < \infty), \quad y \in A.$$

where $e_B(\cdot)$ denotes the equilibrium measure of B and $P_\nu = \sum_x \nu(x) P_x$.

(ii) Deduce from (i) that $A \subset B$ implies $\operatorname{cap}(A) \leqslant \operatorname{cap}(B)$.

(iii) If $A \neq B$ is the inequality in (ii) strict? Justify your answer.

2 Let $\Gamma = (G, \mu, \kappa)$ be a transient weighted graph, P_x denote the law of the associated random walk and $\mathcal{E}(\cdot, \cdot)$ its Dirichlet form. Fix a reference point $0 \in G$ and let

$$h_x = P_x(\tau < \infty), \quad x \in G,$$

where τ is the first time the random walk visits 0.

(a) Define the space H_0.

(b) Prove that $h \in H_0$. *Hint:* approximate h suitably.

(c) Express $\mathcal{E}(h, h)$ solely in terms of the Green's function $g(\cdot, \cdot)$.

(d) Let $\varphi = (\varphi_x)_{x \in G}$ denote the Gaussian free field associated to Γ, with law $\mathbb{P}$. Determine the law of $\psi = (\psi_x)_{x \in G}$ under $\mathbb{P}$, where

$$\psi_x = \varphi_x - h_x \varphi_0.$$

3 Let $(V_x)_{x \in \mathbb{Z}^d}$ and $(W_x)_{x \in \mathbb{Z}^d}$ be two independent families of i.i.d. $N(0, 1)$ Gaussian random variables. Define

$$Y_x = V_x + \frac{1}{2d} \sum_{y \sim x} W_y, \quad x \in \mathbb{Z}^d,$$

where the sum ranges over the $(2d)$ neighbors of x. Let $a_c = a_c(d)$ denote the critical threshold for percolation of the set $\{Y \geqslant a\} = \{x \in \mathbb{Z}^d : Y_x \geqslant a\}$, as $a \in \mathbb{R}$ varies.

(a) For which $x, x' \in \mathbb{Z}^d$ are Y_x and $Y_{x'}$ independent? Justify your answer.

(b) Show that $a_c(d) < \infty$ for all $d \geqslant 2$.

(c) Let $A_R(a)$ denote the event that 0 is connected to distance R in $\{Y \geqslant a\}$ and $\theta_R = \theta_R(a)$ its probability. Show that for all $R \geqslant 1$ and $a \in \mathbb{R}$,

$$-\frac{d}{da}\theta_R \geqslant \sum_{x \in B_R} E\big[V_x 1_{A_R(a)}\big].$$

(d) Deduce from (c) and the OSSS inequality that for some $c = c(d) > 0$,

$$-\frac{d}{da}\theta_R \geqslant c \frac{\theta_R(1-\theta_R)}{\frac{1}{R}\sum_{k=1}^{R} \theta_k}, \quad R \geqslant 1, \, a \in \mathbb{R}.$$

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