UNIVERSITY OF CAMBRIDGE

MAMA/217, NST3AS/217, MAAS/217

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 9:00 am to 11:00 am

PAPER 217

GAUSSIAN PROCESSES AND MEASURES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(X(t) : t \in T)$ be a centred Gaussian process indexed by some set T and let V be a cylindrically measurable subspace of $\mathbb{R}^T$. Show that $\Pr(X \in V) \in \{0, 1\}$.

Now let $T = \mathbb{N}$ and consider jointly Gaussian random variables $X(t) = g_t \sim N(0, \sqrt{t})$. Construct infinite-dimensional linear subspaces V, W of $\mathbb{R}^{\mathbb{N}}$ for which $\Pr(X \in V) = 1$ and $\Pr(X \in W) = 0$, respectively, and justify your answer.

[You may use standard properties of finite-dimensional Gaussian distributions without proof.]

2 Gaussian Processes and Measures

Let $(V, \|\cdot\|)$ be a normed vector space, let $X_1, \ldots, X_n$, be independent Borel random variables taking values in V, and write $S_k = X_1 + \cdots + X_k$ for $1 \leqslant k \leqslant n$. Suppose the X_i are symmetric, that is, the laws of X_i and $-X_i$ are identical. Show that for all $t > 0$,

$$\Pr\Big(\max_{1 \leqslant k \leqslant n} \|S_k\| > t\Big) \leqslant 2 \Pr(\|S_n\| > t).$$

Now let V be a Hilbert space and let (g_j) be an infinite sequence of iid $N(0,1)$ variables. Let further $(u_j : j \in \mathbb{N})$ be a sequence in V and such that $\sum_j \|u_j\|_V^2 < \infty$. Prove that the series $\sum_j g_j u_j$ converges almost surely in V.

[Hint: You may use without proof that a sequence of random variables Z_n converges almost surely to some limit Z in V if $\Pr(\sup_{j \geqslant n} \|Z_j - Z_n\|_V > \epsilon) \to 0$ for every $\epsilon > 0$ as $n \to \infty$.]

3

Let $(X(t) : t \in \mathbb{R})$ be a centred stationary Gaussian process with covariance $\Phi(s,t) = K(s-t), s, t \in \mathbb{R}$, where K has Fourier transform

$$\hat{K}(u) = \int_{\mathbb{R}} e^{iux} K(x) dx = (1 + |u|^2)^{-1}, \; u \in \mathbb{R}.$$

Show that its reproducing kernel Hilbert space (RKHS) equals the Sobolev space $H^1(\mathbb{R})$, and that its law defines a Gaussian Borel probability measure on the space $L^2(\mathbb{R}, \nu)$ for any finite measure ν on $\mathbb{R}$.

[You may use without proof results from Fourier analysis, as well as general theorems on Gaussian processes and RKHS from lectures, provided they are clearly stated.]

4

Denote by $H = L^2((0,1), dx)$ the Hilbert space of square integrable functions on $(0,1)$ for Lebesgue measure dx. Construct a centred Gaussian process $(X(t) : t \in (0,1))$ with the following properties:

1) the law μ_X of X on $\mathbb{R}^{[0,1]}$ satisfies $\mu_X(H) = 1$,

2) X has a version that is almost surely infinitely differentiable on $(0,1)$,

3) $\mu_X(V) = 0$ for every *finite-dimensional* linear subspace V of H.

[You may use without proof the Sobolev embedding theorem, as well as standard properties of trigonometric polynomials.]

END OF PAPER