UNIVERSITY OF CAMBRIDGE

MAMA/215, NST3AS/215, MAAS/215

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 10 June 2026 9:00am to 11:00am

PAPER 215

MIXING TIMES OF MARKOV CHAINS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For a positive integer n, let a *bishop's random walk* be a Markov chain on white squares of the $(2n+1) \times (2n+1)$ chessboard (where squares alternate between black and white horizontally and vertically), in which the top right square is white, defined as follows. At each step, the chain picks a uniformly random square out of all squares with which it shares at least one diagonal and moves to it (it can also pick a square on which it currently is). In other words, it picks a uniform location out of all possible locations to which a standard bishop in chess could move and its own location, and moves to it.

(a) Explain why the bishop's random walk is an irreducible chain on white squares of the chessboard.

(b) Show that the mixing time of the bishop's random walk is of constant order.

[You can use results from the course relating mixing times with coalescence/coupling times as long as they are clearly stated. Hint: It might be useful to consider a diamond-shaped middle region of the chessboard.]

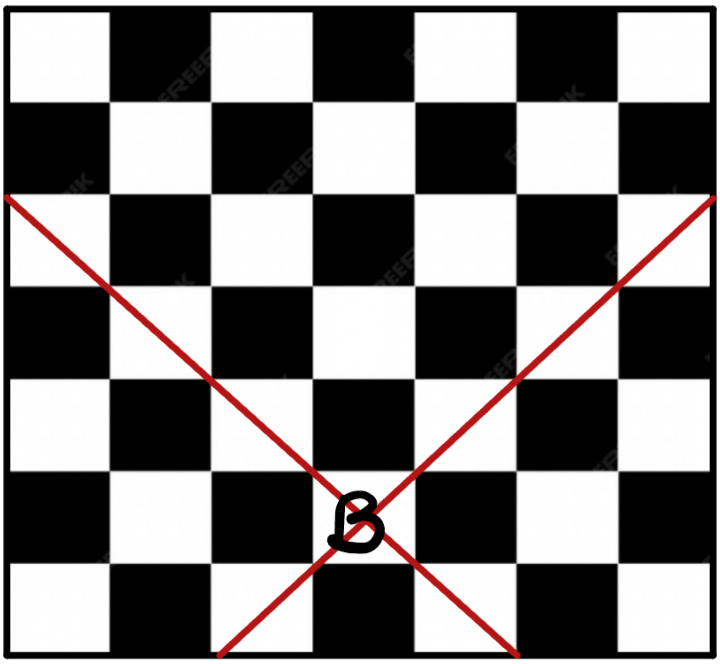

Figure 1: In this example with $2n+1 = 7$, the bishop on the square labelled with B can move to any of the 9 white squares through which at least one of the red lines passes.

2 (a) List the eigenvalues and a basis of eigenfunctions for a lazy simple random walk on the n-dimensional hypercube. For $\varepsilon \in (0,1)$, show that there exists a constant $C(\varepsilon)$ such that the mixing time of this lazy simple random walk satisfies $t_{\mathrm{mix}}(\varepsilon) \leqslant \frac{1}{2} n \log n + C(\varepsilon) n$.

[You may use that for a finite, transitive, irreducible and reversible Markov chain we have $4\|P^t(x,\cdot) - \pi\|_{\mathrm{TV}}^2 \leqslant \sum_{i=2}^{m} \lambda_i^{2t}$ where $1 = \lambda_1 > \lambda_2 \geqslant \ldots \geqslant \lambda_m$ are the eigenvalues of P.]

(b) Show that if a function $f : S \to \mathbb{R}$ and two distributions, μ and ν on S satisfy for some $r, \sigma > 0$ that $|\mu(f) - \nu(f)| = r\sigma$ and $\max\{\mathrm{Var}_\mu(f), \mathrm{Var}_\nu(f)\} \leqslant \sigma^2$ then $\|\mu - \nu\|_{\mathrm{TV}} \geqslant 1 - \frac{8}{r^2}$.

By considering the function $\Phi : \{0,1\}^n \to \mathbb{R}$ defined by $\Phi((x_1, \ldots, x_n)) = n - 2\sum_{i=1}^{n} x_i$, or otherwise, show that the lazy simple random walk on n-dimensional hypercube satisfies that for all $\varepsilon \in (0,1)$, there exists a constant $C(\varepsilon)$ such that $t_{\mathrm{mix}}(\varepsilon) \geqslant \frac{1}{2} n \log n - C(\varepsilon) n$.

[You may use that if the lazy walk starts from $(0, \ldots, 0)$, the variance of Φ applied to its location after t steps is at most n for all $t \geqslant 0$ and that $\mathrm{Var}_\pi(\Phi) = n$.]

(c) Consider an urn model where n balls are moved between the two urns as follows. At each time step, a ball is picked uniformly at random. The ball is returned to the urn it was in with probability $\frac{1}{2}$ and moved to the other urn with probability $\frac{1}{2}$. Let $X_t^{(n)}$ be the number of balls in the first urn at time t. Does the sequence $\left(X^{(n)}\right)_{n \geqslant 1}$ exhibit cutoff?

3 Let P be an irreducible Markov chain on finite state space S with invariant distribution π with minimal value $\pi_{\min}$.

(a) (i) For functions $f, g : S \to \mathbb{R}$ define the Dirichlet form $\mathcal{E}(f,g)$ corresponding to P.

(ii) For reversible P, define the relaxation time t_{rel}. Let $\Lambda(r)$ for $r \geqslant \pi_{\min}$ be the spectral profile of P. State the variational characterisation of the spectral gap and deduce that if P is also lazy then $t_{\text{rel}} \geqslant \frac{1}{\Lambda(r)}$ for all $r \geqslant \pi_{\min}$.

(b) Let α, c and Δ be fixed positive constants. We say that a sequence of graphs $G_n = (V_n, E_n)$ is a (Δ, α, c)-*small set expander* sequence if for all n we have that the maximal degree in G_n is at most Δ, that the isoperimetric profile for lazy simple random walk on G_n satisfies $\Phi_*(c) \geqslant \alpha$, and that $|V_n| \to \infty$ as $n \to \infty$. Show that the lazy simple random walk on such G_n satisfies for $\varepsilon \in (0,1)$ that $t_{\text{mix}}(\varepsilon) \lesssim \log |V_n| + t_{\text{rel}} \log(1/\varepsilon)$.

(c) Let α and Δ be fixed consatnts. We say that the sequence of graphs G_n is a (Δ, α)-*expander* if it is a $(\Delta, \alpha, 1/2)$-small set expander. Let $G_n = (V_n, E_n)$ be a (Δ, α)-expander with $|V_n| = n$ and let H_n be a connected graph on n vertices with degrees at most Δ. Consider a graph M_n obtained by adding n edges between G_n and H_n in such a way that each vertex in G_n is connected to exactly one vertex in H_n. Show that the mixing time of lazy simple random walk on M_n satisfies that $t_{\text{mix}} \lesssim \log n$.

[In this question you are allowed to use without proof that if P is lazy then for any $\varepsilon > 0$ the mixing time of P satisfies $t_{\text{mix}}(\varepsilon) \leqslant 2 \left\lceil \int_{4\pi_{\min}}^{4/\varepsilon} \frac{2du}{u\Lambda(u)} \right\rceil$, where for $r \geqslant \pi_{\min}$, we define the spectral profile $\Lambda(r) = \inf_{A : \pi_{\min} \leqslant \pi(A) \leqslant r} \lambda(A)$. Here, $\lambda(A) = \inf_{f \in c_0^+(A)} \frac{\mathcal{E}(f,f)}{\operatorname{Var}_\pi(f)}$ and $c_0^+(A)$ is a set of non-constantconstant functions which are non-negative on A and 0 on $S \setminus A$. You are also allowed to use without proof the generalised Cheeger's inequality stating for $r \leqslant \frac{1}{2}$ that $\frac{\Phi_*^2(r)}{2} \leqslant \Lambda(r) \leqslant \frac{\Phi_*(r)}{1-r}$ and Cheeger's inequality saying that for a reversible chain we also have $\frac{\Phi_*(1/2)^2}{2} \leqslant \gamma \leqslant 2\Phi_*(1/2)$, where $\Phi_*(r)$ is the isoperimetric profile and γ is the spectral gap.]

4 (a) (i) A *length function* ℓ on a graph $G = (E, V)$ is a function on edges of this graph taking values greater than or equal to 1. Define the path metric corresponding to a length function ℓ.

(ii) For a path metric ρ on $G = (V, E)$, corresponding to a length function ℓ, the *transportation metric* is defined by

$$\rho_K(\mu, \nu) = \inf\{\mathbb{E}[\rho(X, Y)] : (X, Y) \text{ is a coupling of } \mu \text{ and } \nu\}.$$

Let P be the transition matrix of a Markov chain on V. Assume that there is a constant $\alpha > 0$ such that for any two distributions μ and ν on V we have that $\rho_K(\mu P, \nu P) \leqslant e^{-\alpha}\rho_K(\mu, \nu)$. Show that for $\varepsilon \in (0, 1)$ the mixing time of P satisfies

$$t_{\mathrm{mix}}(\varepsilon) \leqslant \frac{1}{\alpha}(\log(\max_{x,y \in V} \rho(x, y)) - \log(\varepsilon)).$$

(b) For a graph $G = (V, E)$ we say that a set of vertices $X \subset V$ is an independent set if no two vertices in X are neighbours in G. Let $|V| = n$, let G have maximal degree at most Δ and let $s \leqslant \frac{n}{3(\Delta+1)}$ be a positive integer. We define P to be a Markov chain on the set S of all independent sets of G of size s, which has transitions as follows. For $X \in S$, pick a vertex $x \in X$ uniformly at random and pick, independently, a vertex $v \in V$ uniformly at random, and let $Y = X \cup \{v\} \setminus \{x\}$. If $Y \in S$, move the chain to Y and otherwise keep it at X.

(i) Check that this chain is irreducible.

(ii) Check that the invariant distribution is uniform on S.

(iii) Show that this chain satisfies $t_{\mathrm{mix}}(\varepsilon) \lesssim sn \log(s/\varepsilon)$.

[In this question you are allowed to use the path coupling technique without proof, i.e. that if Markov chain P takes values in V and if there is a constant α such that for any two neighbours x and y in G there exists a coupling (X_1, Y_1) of $P(x, \cdot)$ and $P(y, \cdot)$ for which

$$\mathbb{E}[\rho(X_1, Y_1)] \leqslant e^{-\alpha}\rho(x, y)$$

then for any distributions μ and ν on V we have that

$$\rho_K(\mu P, \nu P) \leqslant e^{-\alpha}\rho_K(\mu, \nu).$$

You are also allowed to use the existence of an optimal coupling for the total variation distance without proof.]

END OF PAPER