MAMA/212, NST3AS/212, MAAS/212

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 9:00am to 11:00am

PAPER 212

RANDOM DISCRETE STRUCTURES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider bond percolation on $\mathbb{Z}^2$ with probability $p = 1/2$. Let $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ for all $n \in \mathbb{N}$. For $x, y \in \mathbb{Z}^2$ we write $\{x \longleftrightarrow y\}$ for the event that x and y are connected via an open path of edges.

(a) Let $\mathrm{LR}(\ell)$ be the event that there exists an open crossing from the left side of $[0, \ell + 1] \times [0, \ell]$ to its right side, i.e. from $\{0\} \times [0, \ell]$ to $\{\ell + 1\} \times [0, \ell]$. Show that for all $\ell \in \mathbb{N}$ we have

$$\mathbb{P}_{\frac{1}{2}}(\mathrm{LR}(\ell)) = \frac{1}{2}.$$

(b) Let $\partial\Lambda_n = \{y \in \Lambda_n : y_1 \in \{-n, n\} \text{ or } y_2 \in \{-n, n\}\}$. Prove that there exist positive constants c_1 and c_2 so that for all $n \in \mathbb{N}$ and all $x \in \partial\Lambda_n$

$$c_1(\mathbb{P}_{\frac{1}{2}}(0 \longleftrightarrow \partial\Lambda_n))^2 \leqslant \mathbb{P}_{\frac{1}{2}}(0 \longleftrightarrow x) \leqslant c_2(\mathbb{P}_{\frac{1}{2}}(0 \longleftrightarrow \partial\Lambda_n))^2,$$

where we write $\{0 \longleftrightarrow \partial\Lambda_n\} = \cup_{x \in \partial\Lambda_n}\{0 \longleftrightarrow x\}$.

[Hint: You may use the RSW (Russo Seymour Welsh) theorem without proof. You may first wish to show that there exists a positive constant c so that for all n

$$\mathbb{P}_{1/2}(0 \longleftrightarrow \partial\Lambda_n) \leqslant c \cdot \mathbb{P}_{1/2}(0 \longleftrightarrow \partial\Lambda_{2n})\,.]$$

2 Let G be an infinite graph and let o be a vertex of G. Let ω be a p-bond percolation configuration on G. Let μ be a probability measure on infinite directed paths from o that visit any vertex at most a finite number of times. For $r \geqslant 0$ let $B(o,r)$ be the ball of radius r around o in the graph distance of G. For fixed r, we think of the complement of $B(o,r)$ as being represented by a single vertex z_r. For a directed edge $e=(x,y)$ in $B(o,r)$ we let

$$\theta_\omega(e) = \sum_{\substack{\gamma: o \to z_r \\ \gamma \subseteq B(o,r)}} \mathbf{1}(\gamma \text{ open})(\mathbf{1}((x,y)\in\gamma) - \mathbf{1}((y,x)\in\gamma))\frac{\mu(\gamma)}{\mathbb{P}(\gamma \text{ open})},$$

where $\mu(\gamma) = \mu(\{\xi : \xi_{|_{B(o,r)}} = \gamma\})$ and $\{\gamma \text{ open}\}$ means that all edges on γ are open.

(a) Show that θ_ω defines a flow from o to z_r for all ω.

(b) Let $X_r(\omega)$ be the strength of θ_ω. Show that

$$\mathbb{E}[X_r(\omega)] = 1.$$

(c) Suppose that μ has the following property: there exist positive constants C and ζ with $\zeta < p < 1$ such that if ξ and ξ' are independent with law μ, then

$$\mu \times \mu(|\xi \cap \xi'| \geqslant n) \leqslant C \cdot \zeta^n.$$

(i) Show that $\mathbb{E}[(X_r(\omega))^2] \leqslant C\frac{\zeta}{p-\zeta}$.

(ii) By computing the energy of the flow θ_ω or otherwise, deduce that there exists a transient open cluster almost surely.

[Hint: You may use without proof the Paley-Zygmund inequality: for a nonnegative random variable X and $\theta \in [0,1]$ we have

$$\mathbb{P}(X > \theta\mathbb{E}[X]) \geqslant (1-\theta)^2(\mathbb{E}[X])^2/\mathbb{E}[X^2].]$$

3 Let $G=(V,E)$ be an infinite transient graph.

(a) Define the term *wired uniform spanning forest* of G and then describe Wilson's method rooted at infinity for generating it.

(b) Let $K \in \mathbb{N}$ and $w_1,\ldots,w_K \in V$ be such that with positive probability K independent random walks started from $w_1,\ldots,w_K$ have no pairwise intersections. Show that the number of trees of the wired uniform spanning forest is at least K almost surely.

4 Consider p-bond percolation on $\mathbb{Z}^d$ for $d \geqslant 2$ and $p \in (0,1)$. Let C be the cluster of 0 and write

$$\chi(p) = \mathbb{E}_p[|C|]\,,$$

where $|C|$ stands for the cardinality of C. For this question we suppose that p is such that $\chi(p) < \infty$. We write $\{x \longleftrightarrow y\}$ for the event that x and y are connected via an open path of edges. For all $n \geqslant 0$ we let M_n be the number of vertices in

$$C \cap \partial([-n,n]^d \cap \mathbb{Z}^d) = \{x = (x_1, \ldots, x_d) \in C \cap [-n,n]^d : |x| = n\},$$

where $|x| = \sum_{i=1}^d |x_i|$ for $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$.

(a) Show that there exists $m \geqslant 1$ so that

$$\mathbb{E}_p[M_m] \leqslant (1 - \chi(p)^{-1})^m.$$

(b) Show that for all $u \in \mathbb{Z}^d$ we have

$$\mathbb{P}_p(0 \longleftrightarrow u) \leqslant (\mathbb{E}_p[M_m])^{\lfloor |u|/m \rfloor}.$$

(c) Show for all $u \in \mathbb{Z}^d$ and all $k \in \mathbb{N}$ we have

$$\mathbb{P}_p(0 \longleftrightarrow ku) \geqslant (\mathbb{P}_p(0 \longleftrightarrow u))^k.$$

(d) Conclude that for all $u \in \mathbb{Z}^d$

$$\mathbb{P}_p(0 \longleftrightarrow u) \leqslant (1 - \chi(p)^{-1})^{|u|}.$$

END OF PAPER