MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 10 June 2026 2:00pm to 5:00pm

PAPER 202

STOCHASTIC CALCULUS WITH APPLICATIONS TO FINANCE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Stochastic Calculus with Applications to Finance

(a) Let X be a continuous local martingale, and let $Z_t^{(a)} = e^{aX_t - a^2 t/2}$ for $t \geqslant 0$ and $a \in \mathbb{R}$. Suppose there exists a real number $A \neq 0$ such that $Z^{(A)}$ is also a local martingale. Show that $Z^{(a)}$ is a true martingale for all a. [Hint: Use Lévy's characterisation of Brownian motion.]

(b) Let X be a continuous local martingale, and suppose its quadratic variation is of the form

$$[X]_t = \int_0^t H_s \, ds$$

where the process H is previsible and strictly positive. Show that there exists a Brownian motion W and a locally W-integrable previsible process K such that

$$X_t = X_0 + \int_0^t K_s \, dW_s$$

(c) Let X be a continuous, adapted process such that

$$M_t^f = f(X_t) - \int_0^t \left(b(X_s) f'(X_s) + \frac{1}{2}\sigma(X_s)^2 f''(X_s) \right) ds$$

is a local martingale for each $f \in C^2$ where the measurable and locally bounded functions b and σ are given. Suppose that $\sigma(x) > 0$ for all x. Show that X is a weak solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

[Hint: Consider the functions $f(x) = x$ and $f(x) = x^2$.]

2 Stochastic Calculus with Applications to Finance

(a) For a continuous adapted process $X = (X(t))_{t \geqslant 0}$, let

$$\|X\| = \mathbb{E}(\sup_{t \geqslant 0} |X(t)|)$$

Let $\mathcal{X}$ be the set of continuous adapted processes X (or more properly, the equivalence classes of indistinguishable continuous adapted processes X) such that $\|X\|$ is finite. Show that $\mathcal{X}$ is complete with respect to the norm $\|\cdot\|$. You may assume that the filtration satisfies the usual conditions.

(b) Let $(X_n)_n$ and X be continuous adapted processes that have the property that there exists an increasing family of stopping times $T_N \uparrow \infty$ such that for each fixed N we have $\|X_n^{T_N} - X^{T_N}\| \to 0$ as $n \to \infty$, where the notation $Y^\tau = (Y(t \wedge \tau))_{t \geqslant 0}$ denotes the process Y stopped at τ. Show that $X_n \to X$ u.c.p. (uniformly on compacts in probability).

(c) Let X be a continuous semimartingale and $(H_n)_n$ a uniformly bounded sequence of previsible processes such that $H_n \to H$ pointwise. Show that $\int H_n \, dX \to \int H \, dX$ u.c.p. [You may use without proof the fact that if $Y_n \to Y$ u.c.p. and $Z_n \to Z$ u.c.p. then $Y_n + Z_n \to Y + Z$. You may also use without proof the inequality $\mathbb{E}(\sup_{t \geqslant 0} M(t)^2) \leqslant 4 \, \mathbb{E}([M](\infty))$ for a continuous local martingale M with $M(0) = 0$, where $[M]$ is its quadratic variation.]

(d) Let X be a continuous semimartingale. Show that the process $|X|$ is a continuous semimartingale. [Hint: $\frac{1}{n} \log(\cosh(nx)) \to |x|$ uniformly and $\tanh(nx) \to \mathrm{sign}(x)$. You may also find the following facts useful: $\frac{d}{dx} \log \cosh x = \tanh x$, $|\tanh x| < 1$ for all x and $\frac{d}{dx} \tanh x = (\cosh x)^{-2}$.]

3 Stochastic Calculus with Applications to Finance

For a continuous function $f : [0,1] \to \mathbb{R}$ let

$$\|f\| = \sup_N \sum_{k=1}^{2^N} |f(k2^{-N}) - f((k-1)2^{-N})|.$$

(a) For fixed $0 \leqslant t \leqslant 1$, let $f_t(s) = f(s \wedge t)$ for all $0 \leqslant s \leqslant 1$. Assuming $\|f\| < \infty$, show that

$$\|f_{t_2}\| - \|f_{t_1}\| \geqslant |f(t_2) - f(t_1)|$$

for all $0 \leqslant t_1 < t_2 \leqslant 1$.

(b) Show that

$$\|f\| = \sup_{0=t_0<t_1<\cdots<t_{n-1}<t_n=1} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|.$$

(c) Let f be continuously differentiable. Show that $\|f\| < \infty$.

(d) Let $f(0) = 0$ and

$$f(t) = t \sin(1/t) \text{ for } t > 0.$$

Show that $\|f\| = \infty$.

4 Stochastic Calculus with Applications to Finance

Let $S_0 > 0$ be constant, let M be a continuous local martingale with $M_0 = \log S_0$, and let S be the local martingale defined by $S = e^{M - \frac{1}{2}[M]}$. Let $T > 0$ be non-random.

(a)(i) If $[M]_T \geqslant a$ a.s. for a constant $a \geqslant 0$, show that

$$\mathbb{E}(\sqrt{S_T}) \leqslant \sqrt{S_0} e^{-a/8}.$$

(ii) If $[M]_T \leqslant b$ a.s. for a constant $b \geqslant 0$, show that

$$\mathbb{E}(\sqrt{S_T}) \geqslant \sqrt{S_0} e^{-b/8}.$$

Consider a market model with zero interest rate $r_t = 0$ for all $t \geqslant 0$ and a risky asset with time-t price S_t. The market also has a European contingent claim with maturity date T and time-t price C_t, where

$$C_t = \mathbb{E}\big(\sqrt{S_T} | \mathcal{F}_t\big)$$

for $0 \leqslant t \leqslant T$. Assume the process C is continuous.

(b) Briefly explain why the market consisting of the bank account, the risky asset and the contingent claim has no arbitrage.

(c) Consider the Black–Scholes model with $M = \log S_0 + \sigma W$, where σ is a constant and W is a Brownian motion. Find, with proof, an explicit function $\Delta : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$dC_t = \Delta(t, S_t) dS_t.$$

(d) Assuming $a \leqslant [M]_T \leqslant b$ a.s. show that the Black–Scholes implied volatility $\hat{\sigma}$ for the claim satisfies $a \leqslant T\hat{\sigma}^2 \leqslant b$.

END OF PAPER