MAMA/201, NST3AS/201, MAAS/201, MGM3/29

MAT3

MATHEMATICAL TRIPOS Part III

Friday 5 June 2026 2:00pm to 5:00pm

PAPER 201

ADVANCED PROBABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(A_n : n \geqslant 1)$ be a sequence of events and let X be an integrable random variable. Set $\mathcal{F}_n = \sigma(A_1, \ldots, A_n)$.

(a) Show, without appealing to general results on conditional expectation, that, for each $n \geqslant 1$, there exists a random variable X_n with the following properties:

(i) X_n is $\mathcal{F}_n$-measurable,

(ii) X_n is integrable and $\mathbb{E}(X_n 1_A) = \mathbb{E}(X 1_A)$ for all $A \in \mathcal{F}_n$.

(b) Show that, in a certain sense to be made precise, properties (i) and (ii) characterize X_n uniquely.

(c) Show that $(X_n)_{n \geqslant 1}$ converges almost surely and in L^1 to an integrable random variable. Any general theorems to which you appeal should be clearly stated.

(d) Deduce, without appealing to general results on conditional expectation that, for any countably generated σ-algebra $\mathcal{G} = \sigma(A_n : n \geqslant 1)$, there exists a random variable Y with the following properties:

(i) Y is $\mathcal{G}$-measurable,

(ii) Y is integrable and $\mathbb{E}(Y 1_A) = \mathbb{E}(X 1_A)$ for all $A \in \mathcal{G}$.

2

(a) Let X be an integrable real-valued random variable. Define the cumulant generating function ψ of X and its Legendre transform ψ^*.

(b) State Cramér's theorem on the large deviations of sums of independent identically distributed integrable real-valued random variables.

(c) Prove the upper bound in Cramér's theorem.

(d) Assume that we restrict in Cramér's theorem to the case where ψ is finite on $[0, \infty)$ and its derivative ψ' is a strictly increasing continuous map $[0, \infty) \to [m, \infty)$. Prove the lower bound of the theorem in this case.

3

(a) State Doob's upcrossing inequality.

(b) State and prove the almost sure martingale convergence theorem for a discrete-time martingale.

(c) State the almost sure martingale convergence theorem for a continuous-time martingale.

In Parts (d) and (e) you may use any martingale properties associated with Brownian motion without proof but if you use the transience of Brownian motion in $\mathbb{R}^3$ you should prove it. Write $|x|$ for the Euclidean norm $\sqrt{x_1^2 + x_2^2 + x_3^2}$.

(d) Let $(B_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}^3$ with $|B_0| = 1$. Show that there is a random variable Y such that $|B_t|^{-1} \to Y$ almost surely as $t \to \infty$.

(e) By using Brownian scaling or otherwise, show that $Y = 0$ almost surely.

4

(a) What does it mean to say that a random process $(B_t)_{t \geqslant 0}$ is a Brownian motion?

(b) Set

$$Z = \int_0^1 B_t dt.$$

Find the mean and variance of Z and show that Z is a Gaussian random variable.

(c) State Donsker's invariance principle for a real-valued random walk $(S_n)_{n \geqslant 0}$ with steps of mean 0 and variance 1.

(d) Let $(X_n : n \geqslant 1)$ be a sequence of independent indentically distributed random variables of mean 0 and variance 1. Deduce from Donsker's invariance principle that the following random variables converge weakly as $n \to \infty$ and determine their limiting distributions:

(i) $n^{-1/2} \sum_{k=1}^n X_k$,

(ii) $n^{-3/2} \sum_{k=1}^n \sum_{j=1}^k X_j$.

5 Let D be a bounded domain in $\mathbb{R}^d$ and let $(X_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}^d$ starting from $x \in D$. Set

$$T = \inf\{t \geqslant 0 : X_t \notin D\}.$$

(a) Show that $\mathbb{E}_x(T) < \infty$.

(b) Let u be a continuous function on the closure $\bar{D}$ of D and suppose that u is C^2 in D with $|\Delta u(x)| \leqslant C$ for all $x \in D$ for some constant $C < \infty$. Show that, for $x \in D$,

$$\mathbb{E}_x(u(X_T)) = u(x) + \mathbb{E}_x \int_0^T \tfrac{1}{2} \Delta u(X_t) dt.$$

(c) Suppose that $(u_n : n \geqslant 1)$ is a sequence of continuous functions on $\bar{D}$, all of which are C^2 in D. Assume that $u_n = 0$ on the boundary ∂D and that $|\Delta u_n(x)| \leqslant C$ for all $x \in D$ for all n. Show that if $\Delta u_n(x) \to 0$ as $n \to \infty$ for all $x \in D$ then $u_n(x) \to 0$ as $n \to \infty$ for all $x \in D$.

[*You may assume standard results on one-dimensional Brownian motion and on martingales associated with Brownian motion in* $\mathbb{R}^d$.]

6

(a) What does it mean to say that a real-valued random process is a Lévy process?

(b) Let $(X_t)_{t \geqslant 0}$ be a real-valued Lévy process and suppose that X_1 has finite mean μ and variance $\sigma^2 \in (0, \infty)$. Show that X_t has finite variance for all $t \geqslant 0$ and determine its value.

(c) Suppose now that $\mu = 0$ and set $M_t = X_t^2 - \sigma^2 t$. Show that $(M_t)_{t \geqslant 0}$ is a martingale.

(d) Suppose now further that $(X_t)_{t \geqslant 0}$ has bounded jumps. Fix integers $a, b \geqslant 1$ and set

$$T = \inf\{t \geqslant 0 : X_t \leqslant -a \text{ or } X_t \geqslant b\}.$$

Show that $\mathbb{E}(X_T^2) = \sigma^2 \mathbb{E}(T)$.

(e) Suppose in fact that $(X_t)_{t \geqslant 0}$ is the difference of two independent Poisson processes of the same rate $\lambda > 0$. Find $\mathbb{E}(T)$.

END OF PAPER