UNIVERSITY OF CAMBRIDGE

MAMA/168, NST3AS/168, MAAS/168

MAT3

# MATHEMATICAL TRIPOS Part III

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## PAPER 168

## ANALYSIS OF BOOLEAN FUNCTIONS

### Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (i) Let $f$ be a function defined on the $p$-biased cube $\{-1,1\}^n$. Derive formulae for $\mathbf{I}(f)$ and $\mathbf{Stab}_\rho(f)$ when $\rho \in [0,1]$ in terms of the $p$-biased Fourier coefficients of $f$. Define the *linear weight* of $f$ to be $\sum_i \hat{f}(\{i\})^2$. Prove that $\frac{d}{d\rho}\mathbf{Stab}_\rho(f)$ is equal to the linear weight of $f$ at $\rho = 0$ and $\mathbf{I}(f)$ at $\rho = 1$ (taking right and left derivatives, respectively).

(ii) Show how to use Fourier analysis to prove Arrow's theorem. [You may assume the Fourier formula for $\mathbf{Stab}_\rho$ even when $\rho < 0$.]

**2** (i) Let $f : \{-1,1\}^n \to \mathbb{R}$. Assuming the Bonami lemma and any basic Fourier formulae you might need, prove that $\|T_{1/\sqrt{3}}f\|_4 \leqslant \|f\|_2$, and deduce that $\mathbf{Stab}_{1/3}f \leqslant \|f\|_{4/3}^2$.

(ii) State and prove Friedgut's junta inequality for Boolean functions of approximate degree at most $k$.

(iii) Let $f : \{-1,1\}^n \to \mathbb{R}$ be a function of degree at most $k$ that is not identically zero. Prove that $\mathbb{P}[f(x) \neq 0] \geqslant 2^{-k}$. [Hint: use induction on $n$.]

(iv) Suppose in addition that $f : \{-1,1\}^n \to \{-1,1\}$ (i.e., that $f$ is Boolean). Deduce that $f$ is a $k2^{k-1}$-junta. [Hint: consider the functions $D_i f$.]

**3** (i) What is an $(\epsilon, p, r)$-quasirandom Boolean function $f : \{0,1\}^n \to \{0,1\}$? State a regularity lemma concerning such functions. [You do not need to prove it.]

(ii) Prove that for every $\zeta > 0$ and $\alpha > 0$ there exist $\epsilon > 0$ and $r \in \mathbb{N}$ such that if $p \in [\zeta, \frac{1}{2} - \zeta]$ and $f$ is an $(\epsilon, p, r)$ quasirandom monotone Boolean function with $\mathbb{E}f^{(p)} = \alpha$, then $\mathbb{E}f^{(1/2)} > 1/2$. [You may assume any earlier results from the course that you might need.]

(iii) Deduce that for every $p \in (0, 1/2)$ and every $\epsilon > 0$ there exists $m$ such that if $n$ is a positive integer and $\mathcal{A}$ is an intersecting family of subsets of $n$, then there exists $J \subset \{1, 2, \ldots, n\}$ with $|J| \leqslant m$ and an intersecting family $\mathcal{B}$ of subsets of $J$ such that $\mu_p(\mathcal{A} \setminus \overline{\mathcal{B}}) \leqslant \epsilon$, where $\overline{\mathcal{B}}$ is the upwards closure of $\mathcal{B}$.

(iv) Fix a positive integer $k$, let $\mathcal{A} \subset [n]^{(k)}$ be an intersecting family of subsets of $[n]$ and let $\epsilon > 0$. Prove that either $|\mathcal{A}| = o(\binom{n-1}{k-1})$ or there exist $A, B \in \mathcal{A}$ with $|A \cap B| = 1$. Deduce that if $n$ is sufficiently large, then there exists $i$ such that all but at most $\epsilon\binom{n-1}{k-1}$ sets in $\mathcal{A}$ contain $i$.

**4** (i) Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}X_i^4 \leqslant C$ for every $i$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multilinear function of degree at most $k$. Prove that

$$\mathbb{E}f(X_1, \ldots, X_n)^4 \leqslant A^{2k}(\mathbb{E}f(X_1, \ldots, X_n)^2)^2,$$

for some constant $A$ that depends on $C$ only. Prove also that if in addition we assume that $\mathbb{E}X_i^3 = 0$ for each $i$ and that $C \leqslant 9$, then we can take $A = 3$.

(ii) State and prove the invariance principle for degree-$k$ functions.

(iii) Let $A = (a_{ij})$ be an $n \times n$ matrix with zeros on the diagonal and suppose that the sum of squares of the entries in each row and column is at most 1. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function with $\|\psi''''\|_\infty \leqslant M$. Let $X_1, \ldots, X_n$ be independent random variables uniformly distributed in $\{-1, 1\}$ and let $Y_1, \ldots, Y_n$ be independent standard Gaussians. Prove that

$$\Big|\mathbb{E}\psi\Big(\sum_{i,j} a_{ij}X_iX_j\Big) - \mathbb{E}\psi\Big(\sum_{i,j} a_{ij}Y_iY_j\Big)\Big| \leqslant 27Mn.$$

## END OF PAPER