UNIVERSITY OF CAMBRIDGE

MAMA/166, NST3AS/166, MAAS/166

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 5:00pm

PAPER 166

DIOPHANTINE ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Question 1 is worth 30 marks.
Questions 2 and 3 are each worth 35 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Prove that for every real irrational number α and $N \in \mathbb{Z}_{>0}$, there is $p/q \in \mathbb{Q}$ with $1 \leqslant q \leqslant N$ such that

$$|\alpha - p/q| \leqslant \frac{1}{qN}.$$

(b) State a version of the subspace theorem.

(c) State and prove Roth's theorem on rational approximation of algebraic numbers using the subspace theorem.

(d) Let α be a real irrational number. We say that a rational number p/q is a good approximation for α if $\gcd(p,q) = 1$, $q \geqslant 1$ and $|\alpha - \widetilde{p}/\widetilde{q}| \geqslant |\alpha - p/q|$ for all rational $\widetilde{p}/\widetilde{q}$ with $1 \leqslant \widetilde{q} \leqslant q$. We define $(p_n/q_n)_n$ to be the sequence of all good approximations for α in increasing order of the denominator q_n. Prove that the largest prime factor of q_n diverges to infinity. [*Hint:* You may find each of Parts (a), (b) and (c) useful.]

[*In Part (b), any version of the subspace theorem stated in the lectures will be accepted. If you wish to use a different version in Part (c) or (d), then you need to include the statement of this other version, as well.*]

2 (a) Define the *Mahler measure* of a polynomial, and state, without proof, a formula for the height of an algebraic number in terms of the Mahler measure of a certain polynomial.

(b) Let $k \in \mathbb{Z}_{\geqslant 1}$, $n_1, \ldots, n_k \in \mathbb{Z}_{\geqslant 0}$, and let $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$ be two polynomials that are of degree at most n_j in the variable x_j for each j. Let $\alpha_1, \ldots, \alpha_k \in \overline{\mathbb{Q}}$. Prove that

$$H\left(\frac{P(\alpha_1, \ldots, \alpha_k)}{Q(\alpha_1, \ldots, \alpha_k)}\right) \leqslant \max(\mathcal{L}(P), \mathcal{L}(Q)) \prod_{j=1}^{k} H(\alpha_j)^{n_j}.$$

(c) Let $\alpha \in (0,1)$ be an algebraic number of degree $d \in \mathbb{Z}_{\geqslant 2}$. Let $h \in \mathbb{Z}_{\geqslant 2}$, and let

$$A_n = \left\{ \sum_{j=0}^{n-1} a_j \alpha^j : a_j \in \{0, \ldots, h-1\} \text{ for all } j \right\}$$

for $n \in \mathbb{Z}_{>0}$. Prove that $|A_n| \leqslant (Chn)^{d+1} H(\alpha)^{dn} + 1$ for all n, where C is an absolute constant, that is, independent of all the parameters. [The statement is true with $C = 1$, but a proof giving a larger value will be accepted. *Hint:* You may find it useful to give a suitable lower bound on the distance between any two distinct elements of A_n.]

(d) Continuing with the notation and assumptions of Part (c), suppose that d is a prime and let $k, n \in \mathbb{Z}_{>0}$ be such that $H(\alpha)^{kd} > nh$. Prove that the following is true for at least one of $\beta = \alpha^k$ or $\beta = \alpha^{k+1}$: For all non-zero polynomials $P \in \mathbb{Z}[X]$ of degree at most $n-1$ with coefficients less than h in absolute value, we have $P(\beta) \neq 0$.

(e) Continuing with the notation and assumptions of Parts (c) and (d), suppose that $h > H(\alpha)^d$. Prove that

$$|A_n| \geqslant H(\alpha)^{dn \cdot \frac{\log h}{3 \log nh}}$$

for all $n \in \mathbb{Z}_{\geqslant 1}$. [*Hint:* Use Part (d).]

[*In Parts (c)–(e), you may use any results from the lectures about heights and Mahler measure without proof provided you state them clearly.*]

3 In this question, $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$ is a fixed algebraic integer of degree 3, and K is a real quadratic number field, that is, a field $K \subset \mathbb{R}$ with $[K : \mathbb{Q}] = 2$.

(a) Give an explicit constant $c > 0$, depending only on α such that $|\alpha - \beta| \geqslant cH(\beta)^{-6}$ for all $\beta \in K$.

(b) State a theorem giving a lower bound for homogeneous linear forms in logarithms of algebraic numbers with integer coefficients.

(c) Prove that there are effective constants $\varepsilon > 0$ and $c > 0$ depending only on α and K such that

$$|\alpha - \beta| \geqslant cH(\beta)^{-(6-\varepsilon)}$$

for all $\beta \in \mathcal{O}_K$. [*Hint:* Assuming $|\alpha - \beta| \leqslant H(\beta)^{-(6-\varepsilon)}$, begin by giving an upper bound for $\prod_{i=1}^{3} \prod_{j=1}^{2} |\alpha_i - \beta_j|$, where $\alpha_1, \alpha_2, \alpha_3$ are the Galois conjugates of α and β_1, β_2 are the Galois conjugates of β provided $\beta \notin \mathbb{Z}$, then use Part (b) for an appropriate linear form in logarithms.]

[*In this question, you may use any results from the lectures without proof provided you state them clearly.*]

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