UNIVERSITY OF CAMBRIDGE

MAMA/154, NST3AS/154, MAAS/154

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 4 June 2026 9:00am to 12:00pm

PAPER 154

INTRODUCTION TO NONLINEAR ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The energy super critical soliton

We work in dimension $d \geqslant 3$ with radially symmetric real valued functions. For $x \in \mathbb{R}^d$, we let $r = |x| = \sqrt{\sum_{i=1}^d x_i^2}$. We pick $p > 1$ and assume in the whole question the range

$$s_c = \frac{d}{2} - \frac{2}{p-1} \geqslant 1.$$

Our aim is to solve the problem

$$\left| \begin{array}{l} \Delta u + u^p = 0 \\ u(r) > 0 \\ \frac{du}{dr}(r=0) = 0 \\ \lim_{r\to+\infty} u(r) = 0 \end{array} \right. \tag{0.1}$$

1. Rewrite (0.1) as an ode problem and explain why we can always assume $u(0) = 1$.

2. Consider the change of variables $v(s) = r^{\frac{2}{p-1}} u(r)$ with $s = \log r$. Show that v satisfies the ode with $\dot{v} = \frac{dv}{ds}$:
$$\ddot{v} + 2(s_c - 1)\dot{v} - c_\infty^{p-1} v + v^p = 0$$
for some constant $c_\infty(d,p) > 0$ to be determined.

3. Show that the radial ode $\Delta u + u^p = 0$ always admits an homogeneous solution $u(r) = \frac{c}{r^\beta}$ for some parameters c, β to be determined.

4. For $s_c = 1$, give a closed formula for a solution $u(r)$ to (0.1). Show in particular that
$$\lim_{r\to+\infty} r^\alpha u(r) > 0$$
for some $\alpha = \alpha(d)$ to be determined. Does this solution satisfy $\nabla u \in L^2(\mathbb{R}^d)$? You may use without proof that the soliton solution to $Q'' - Q + Q^p = 0$ is
$$Q(x) = \left[\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)} \right]^{\frac{1}{p-1}}.$$

5. We now assume $s_c > 1$. Compute $\dot{J}$ where $J = \frac{\dot{v}^2}{2} + f(v)$ with $f(v) = -\frac{c_\infty^{p-1}}{2} v^2 + \frac{v^{p+1}}{p+1}$ and draw schematically the graph of f on $\mathbb{R}_+$.

6. Show that for $s_c > 1$, (0.1) has a solution which satisfies
$$\lim_{r\to+\infty} r^\alpha u(r) > 0$$
for some $\alpha = \alpha(d,p)$ to be determined. Does this solution satisfy $\nabla u \in L^2(\mathbb{R}^d)$?

2 Blow up for $E_0 \leqslant 0$ for $0 < s_c < 1$

We work in $\mathbb{R}^d$ with $d \geqslant 3$. We consider the focusing

$$(NLS) \quad \left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u(0,x) = u_0(x) \end{array} \right. \quad , \quad x \in \mathbb{R}^d$$

in the range

$$s_c = \frac{d}{2} - \frac{2}{p-1}, \quad 0 < s_c < 1.$$

We let H^1_r be the set of H^1 functions with radial symmetry. The energy functional is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx.$$

We pick $u_0 \in H^1_r \backslash \{0\}$ and assume

$$E(u_0) \leqslant 0.$$

We let $u \in \mathcal{C}^0([0,T), H^1_r)$ be the corresponding solution to (NLS). The aim of this question is to show that the solution must blow up in finite time ie $T < +\infty$ (we do not assume the extra regularity $|x|u_0 \in L^2$ as in class).

1. Compute $\sigma > 0$ such that

$$\forall u \in H^1(\mathbb{R}^d), \quad \|u\|_{L^{p+1}} \lesssim \|u\|_{\dot{H}^\sigma}$$

and prove

$$\frac{2}{p+1} < \sigma < 1.$$

2. Use the conservation laws to prove the existence of a constant $c(u_0) > 0$ such that $\forall t \in [0,T), \quad \|\nabla u(t)\|_{L^2} \geqslant c(u_0)$.

3. Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ with spherical symmetry, prove the formulas:

$$\frac{1}{2}\frac{d}{dt}\int \chi |u|^2 = Im\left(\int \nabla\chi \cdot \nabla u \overline{u}\right),$$

and

$$\frac{1}{2}\frac{d}{dt} Im\left(\int \nabla\chi \cdot \nabla u \overline{u}\right) = \int \chi'' |\nabla u|^2 - \frac{1}{4}\int \Delta^2 \chi |u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \Delta\chi |u|^{p+1}.$$

4. Prove that for all $u \in H^1_r$,

$$\forall R > 0, \quad \|u\|^2_{L^\infty(r \geqslant R)} \leqslant \frac{2}{R^{d-1}}\|u\|_{L^2}\|\nabla u\|_{L^2}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

5. Let $R > 0$, $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ with spherical symmetry and

$$\psi(x) = \left| \begin{array}{l} \frac{|x|^2}{2} \text{ for } |x| \leqslant 2 \\ 0 \text{ for } |x| \geqslant 10 \\ \psi'' \leqslant 1. \end{array} \right.$$

Let

$$\chi(x) = \psi_R(x) = R^2 \psi\left(\frac{x}{R}\right),$$

show that

$$c(d,p) \int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt} Im\left(\int \nabla\psi_R \cdot \nabla u \overline{u}\right) \leqslant C(d,p) \left[\int_{|x| \geqslant 2R} |u|^{p+1} + \frac{1}{R^2}\int_{2R \leqslant |x| \leqslant 10R} |u|^2\right]$$

for some constants $c(d,p), C(d,p) > 0$ independent of R and u.

6. Prove that $T < +\infty$. (hint: observe that $1 < p < 5$)

3 Dirac mass concentration for L^2 critical NLS

We consider the L^2 critical focusing

$$(NLS) \quad \left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^2 = 0 \\ u(0,x) = u_0(x) \end{array} \right. \quad , \quad x \in \mathbb{R}^2.$$

Given a initial data $u_0 \in H^1$, we let $0 < T \leqslant +\infty$ be the maximal life time in H^1. We let H^1_r be the set of H^1 functions with radial symmetry.

1. Give (without proof) the variational characterization of the ground state Q and the associated sharp lower bound on the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{R}^2} |u|^4 dx.$$

2. Let $w \in H^1$ with $\|w\|_{L^2} \leqslant \|Q\|_{L^2}$ and ψ a smooth bounded real valued function, show that

$$\left| Im\left(\int \overline{w}\, \nabla w \cdot \nabla \psi \, dx \right) \right|^2 \leqslant 2E(w) \int |\nabla \psi|^2 |w|^2 \, dx.$$

3. Show that for $u_0 \in H^1$, $\|u_0\|_{L^2} < \|Q\|_{L^2} \Rightarrow T = +\infty$.

4. Show the compactness of the Sobolev embedding $H^1_r \subset L^4$.

5. For the rest of the question, we let $u_0 \in H^1_r$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and assume $T < +\infty$. Show that there exists a sequence $t_n \to T$, $\lambda(t_n) \to 0$ and $\gamma \in \mathbb{R}$ such that

$$v(t_n, x) = \lambda(t_n) u(t_n, \lambda(t_n) x) \to Q e^{i\gamma} \;\text{ in }\; H^1.$$

6. Show that

$$\forall \varepsilon > 0,\ \exists R > 0\ s.t.\ \forall t \in [0, T[,\quad \int_{|x| \geqslant R} |u(t,x)|^2 dx < \varepsilon.$$

7. Show that forall $\phi \in \mathcal{C}_0(\mathbb{R}^2, \mathbb{C})$ bounded,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} |u|^2(t_n, x) \phi(x) dx = \phi(0) \|Q\|^2_{L^2}.$$

END OF PAPER