MAMA/152, NST3AS/152, MAAS/152

MAT3

MATHEMATICAL TRIPOS Part III

Friday 12 June 2026 2:00pm to 5:00pm

PAPER 152

TORIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

(a) Give an example of a singular, proper toric surface. Make sure you justify why your example is singular and proper.

(b) Resolve the singularities of your chosen example via a sequence of toric morphisms.

(c) In the example sheets you have proved that every proper toric surface is projective. Verify this by explicitly finding a very ample line bundle for the smooth variety obtained from your example.

2 *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

Recall that the Hirzebruch surface $\mathbb{F}_k$ is the toric surface associated to the fan Σ with $|\Sigma| = \mathbb{R}^2$ and with rays generated by $(1,0),\ (0,1),\ (-1,k), (0,-1)$.

(a) Compute its Class group, and Euler characteristic;

(b) Describe $\mathbb{F}_k$ as a geometric quotient;

(c) Give an example of a basepoint-free divisor such that the Kodaira map induces the bundle projection $\mathbb{F}_k \to \mathbb{P}^1$.

3 *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

(a) Let $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_t$ be the morphism given by $t \mapsto xy$. Let X be the blowup of $\mathbb{A}^2$ at the origin and consider the composite $f : X \to \mathbb{A}^1$. Describe it in terms of fans;

(b) Using toric methods, prove that the scheme theoretic preimage $X_0 = f^{-1}(0)$ is non-reduced;

(c) Describe an open cover of X_0 by affine schemes, and describe each affine scheme explicitly in terms of equations.

4 *You may use any result covered in lectures, provided you state it clearly. Unless stated otherwise, all cones are strictly convex rational polyhedral, and all toric varieties are normal.*

Recall that a fan is Complete if and only if its corresponding toric variety is proper.

(a) Let Σ be a complete fan in $N_{\mathbb{R}}$ and let τ be a cone in Σ. Recall that we defined the fan $\text{Star}(\tau)$ as

$$\text{Star}(\tau) = \{\bar{\sigma} \subset N(\tau)_{\mathbb{R}} \mid \tau \preceq \sigma\}.$$

Show that $\text{Star}(\tau)$ is a complete fan in $N(\tau)_{\mathbb{R}}$.

(b) Deduce that the minimal number of rays of a complete fan is $dim(N_{\mathbb{R}}) + 1$.

END OF PAPER