UNIVERSITY OF CAMBRIDGE

MAMA/150, NST3AS/150, MAAS/150

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 4 June 2026 2:00pm to 5:00pm

PAPER 150

ANALYTIC NUMBER THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let Λ denote the von Mangoldt function. Prove that $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}$ for all integers $n \geqslant 1$.

[*You may use any properties of the Dirichlet convolution, provided you state them clearly.*]

(b) Prove that for $x \geqslant 2$ we have

$$\sum_{n \leqslant x} \tau(n) = x \log x + Cx + O(x^{1/2}),$$

for some constant C, where τ is the divisor function.

(c) Prove that for any integer $a \geqslant 1$ and any $x \geqslant z \geqslant w \geqslant 2$ we have

$$|\{n \in [1,x] \colon an+1 \text{ has no prime factor in } [w,z]\}| \ll \prod_{p|a} \left(1 - \frac{1}{p}\right)^{-1} \frac{\log w}{\log z} x.$$

[*You may use Mertens' theorem and any results about sieves stated in the course.*]

2

(a) State and prove the Euler product formula for the Dirichlet series of a multiplicative function $f\colon \mathbb{N} \to \mathbb{C}$ with $|f(n)| \leqslant 1$ for all $n \in \mathbb{N}$.

(b) State the functional equation for the Riemann zeta function, and use it to prove that if $\zeta(s)$ has a non-real zero s with $\mathrm{Re}(s) \neq 1/2$, then it has at least one zero s with $\mathrm{Re}(s) \in (1/2, 1]$.

[*You may use the fact that* $\zeta(s)$ *has no zeros with* $\mathrm{Re}(s) > 1$.]

(c) Prove that for every $\varepsilon > 0$ and $C > 0$, there exists a real number $x \geqslant 1$ such that

$$\left|\sum_{n \leqslant x} \mu(n)\right| \geqslant C x^{1/2-\varepsilon}.$$

[*You may assume that the Riemann zeta function has at least one nontrivial zero.*]

(d) State Halász's theorem.

(e) Using Halász's theorem, or otherwise, prove that for any multiplicative function $f\colon \mathbb{N} \to \{-1, +1\}$, for $x \geqslant 1$ we have

$$\min\left\{\left|\sum_{n \leqslant x} f(n)\right|, \left|\sum_{n \leqslant x} f(n)\mu(n)\right|\right\} = o(x).$$

[*You may use any results on the pretentious distance given in the course, provided you state them clearly.*]

3

(a) State Perron's formula.

(b) Prove that there exists a constant A such that, for any $x \geqslant 1$, the residue of the function

$$\left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 \frac{x^s}{s}$$

at $s = 1$ is $x \log x + Ax$.

[*You may assume that* $\frac{\zeta'(s)}{\zeta(s)}$ *has a simple pole at* $s = 1$ *with residue* -1.]

(c) Prove that $(\Lambda * \Lambda)(n) \leqslant (\log n)^2$ for all integers $n \geqslant 1$ and that

$$\sum_{n=1}^{\infty} \frac{(\log n)^2}{n^{1+u}} \ll u^{-3}$$

for all $u \in (0, 1]$.

(d) Prove that there exists a constant $c > 0$ such that for all $x \geqslant 2$ we have

$$\sum_{n \leqslant x} (\Lambda * \Lambda)(n) = x \log x + Ax + O(x \exp(-c\sqrt{\log x})),$$

where A is the constant from part (b).

[*You may assume a zero-free region for the Riemann zeta function and bounds for* $\zeta'(s)/\zeta(s)$ *inside this region, provided you state them clearly.*]

END OF PAPER