MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 2:00pm to 4:00pm

PAPER 145

ALGEBRAIC METHODS IN COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State and prove the matrix-tree theorem for directed loopless weighted graphs.

(b) Hence, or otherwise, show that given any positive integer $n \geqslant 3$, the directed graph on n vertices $v_1, \ldots, v_n$ with each v_i having two out-edges, one to v_{i+1} and one to v_{i+2} with subscripts taken mod n (so v_{n-1} has out-edges to v_n and v_1, and v_n has out-edges to v_1 and v_2), has exactly $\frac{1}{3}(2^n - (-1)^n)$ directed spanning trees rooted towards v_n.

[A directed spanning tree rooted towards v_n is a connected acyclic subgraph containing no out-edges from v_n and precisely one out-edge from each of $v_1, \ldots, v_{n-1}$.]

2 (a) Define the (unsigned) Stirling number of the second kind $\left\{ {n \atop k} \right\}$ and write down its recurrence relation.

(b) Show that

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = x^n$$

as a polynomial identity in x. *[Recall that $x^{\underline{k}}$ denotes $x(x-1)\dots(x-k+1)$.]*

An *independent set* of a graph is defined to be a subset of its vertices with no edges between them. Given an (undirected) graph G on n vertices, let $\left\{ {n \atop k} \right\}_G$ denote the number of partitions of the vertices of G into k independent sets.

(For example, if G is the empty graph E_n on n vertices, $\left\{ {n \atop k} \right\}_{E_n} = \left\{ {n \atop k} \right\}$.)

Given a positive integer $n \geqslant 2$, let P_n denote the *path* on n vertices $v_1, \dots, v_n$ with edge set $\{v_i v_{i+1}, 1 \leqslant i \leqslant n-1\}$, and let C_n denote the *n-cycle* defined for $n \geqslant 3$ by the addition of an edge $v_n v_1$ to P_n, and defined for $n = 2$ by $C_2 = P_2$. Let $n \geqslant 2$ be fixed.

(c) Show that

$$\left\{ {n \atop k} \right\}_{P_n} = \left\{ {n-1 \atop k-1} \right\}.$$

(d) Find a closed formula in terms of x and n for

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\}_{P_n} x^{\underline{k}}.$$

(e) Show that

$$\left\{ {n \atop k} \right\}_{P_n} = \left\{ {n \atop k} \right\}_{C_n} + \left\{ {n-1 \atop k} \right\}_{C_{n-1}}$$

and deduce that

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\}_{C_n} x^{\underline{k}} = (x-1)^n + (-1)^n (x-1).$$

(f) Let x be a fixed positive integer. Show that for any graph G and any set of x distinct colours, the expression

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\}_G x^{\underline{k}}$$

counts the number of ways of colouring the vertices of G in such a way that the vertices of each colour form an independent set.

3 A *walk* from $(0,0)$ to $(2n,0)$ in $\mathbb{Z}^2$ is a sequence in $\mathbb{Z}^2$ beginning with $(0,0)$ and ending with $(2n,0)$. The differences between consecutive terms of this sequence are called the *steps* of this walk. A walk is said to *lie above* the x-axis if for every (x,y) on this walk, $y \geqslant 0$. It is said to *lie strictly above* the x-axis if for every (x,y) on this walk except for $(0,0)$ and $(2n,0)$, $y > 0$. The definitions for a walk to *lie below* and *lie strictly below* are analogous.

(a) Let C_n count the number of walks in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ which lie above the x-axis all of whose steps are $(1,1)$ or $(1,-1)$. Explain why the generating function $C(x) = \sum_{n\geqslant 0} C_n x^n$ satisfies

$$C(x) = 1 + xC(x)^2$$

and solve to find a closed form expression for $C(x)$.

(b) Let D_n count the number of walks in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ which lie strictly above the x-axis, all of whose steps are $(1,1)$ or $(1,-1)$. Show that $D_n = C_{n-1}$ and deduce that $D(x) = \sum_{n\geqslant 1} D_n x^n = xC(x)$.

(c) Call a step between (x,y) and (x',y') of a walk *negative* if both $y, y' \leqslant 0$. By introducing a weight function $\sqrt{t}$ on negative steps, verify that $E(x) = \sum_{n\geqslant 1} D_n (tx)^n$ is the generating function counting the sum of weights of all walks from $(0,0)$ to $(2n,0)$ which lie strictly below the x-axis. Explain why

$$\frac{1}{1-(D(x)+E(x))} = \sum_{n\geqslant 0} f(t,n)x^n,$$

where $f(1,n)$ counts the number of all walks in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ taking steps $(1,1)$ or $(1,-1)$, and where the coefficient of t^k in $f(t,n)$ is the number of such walks with exactly $2k$ negative steps.

(d) Deduce that

$$\sum_{n\geqslant 0} f(t,n)x^n = \frac{2}{\sqrt{1-4x}+\sqrt{1-4tx}} = \sum_{n\geqslant 0} C_n x^n \left[\frac{1-t^{n+1}}{1-t}\right]$$

and explain why the number of walks in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ taking steps $(1,1)$ or $(1,-1)$ with exactly $2k$ negative steps is independent of k, $0 \leqslant k \leqslant n$.

4 (a) Let $\{S_1,\ldots,S_m\}$ be a family of subsets of $[n]$ such that $|S_i|$ is odd for all i and $|S_i \cap S_j|$ is even for all $i \neq j$. Prove that $m \leqslant n$.

(b) State the Frankl-Wilson Theorem.

Let $\{S_1,\ldots,S_m\}$ be a family of subsets of $[n]$ such that for some prime p and some positive integers $r \geqslant 1$ and $k \geqslant 2$, the size of each subset $|S_i|$ is not divisible by p^r, but $|S_{i_1} \cap \cdots \cap S_{i_k}|$ is divisible by p^r for all distinct $i_1,\ldots,i_k$. Prove that $m \leqslant (k-1)(n+1)$.

Assume now that $r = 1$. Does there exist such a family of size exactly $(k-1)(n+1)$?

5 (a) State the Alon-Tarsi Lemma.

Hence, or otherwise, deduce some version of the Combinatorial Nullstellensatz.

(b) Hence, or otherwise, show that for any finite field $\mathbb{F}_q$ and any positive integer n, if $H_1, \ldots, H_m$ is a family of affine hyperplanes in $\mathbb{F}_q^n$, none of which contain $(0, \ldots, 0)$, but whose union contains every other point $(x_1, \ldots, x_n)$ except $(0, \ldots, 0)$, then $m \geqslant (q-1)n$.

END OF PAPER