UNIVERSITY OF CAMBRIDGE

MAMA/144, NST3AS/144, MAAS/144

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 16 June 9:00am to 11:00am

PAPER 144

MODEL THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{L} = \{E\}$, E a binary relation symbol.

Let T_1 be the theory that expresses that E is an equivalence relation that has exactly two infinite equivalence classes.

Let T_2 be the theory that expresses that E is an equivalence relation such that for every $n \in \mathbb{N}$, E has infinitely many equivalence classes of size n.

(*a*) For which κ (if any) is T_1 κ-categorical? Justify your answer.

(*b*) For which κ (if any) is T_2 κ-categorical? Justify your answer.

(*c*) Show that T_1 has quantifier elimination.

(*d*) Explain why T_2 does not have quantifier elimination. Expand the language by additional relations such that T_2 has quantifier elimination in the expanded language. *[You do not need to justify your choice.]*

2 For the i^{th} prime p_i, let $\mathbb{F}_i$ be the field with p_i elements, in the language $\mathcal{L}_{ring}$ of rings.

(*a*) Describe an ultrafilter $\mathcal{U}_1$ on ω such that $\prod_{i\in\omega} \mathbb{F}_i/\mathcal{U}_1$ has characteristic p_i. Justify your answer.

(*b*) Show for a non-principal ultrafilter $\mathcal{U}_2$ on ω that $\mathcal{F} = \prod_{i\in\omega} \mathbb{F}_i/\mathcal{U}_2$ has a unique algebraic extension of degree n for every $n \in \omega$.

[Hint: Recall that an algebraic extension of degree n of a field F is given by $F(\alpha)$ where α is a solution to an irreducible polynomial of degree n with coefficients in F. You may use the fact that any finite field has a unique algebraic extension of degree n for every $n \in \omega$, as well as any other facts from finite field theory.]

3 Let $\mathcal{L}$ be a countable language.

(*a*) Let T be a complete theory. Define what it means for $\mathcal{M} \models T$ to be $\aleph_0$-*homogeneous*.

(*b*) Let T be a complete theory. Show that every countable model of T has a countable $\aleph_0$-homogeneous elementary extension.

(*c*) Find a countable model of $\mathrm{Th}(\mathbb{Z}, <)$ that is *not* $\aleph_0$-homogeneous. *Justify your answer.*

4 Let T be a complete $\mathcal{L}$-theory, let $\mathcal{M} = (M, \mathcal{I}) \models T$, and suppose that T is strongly minimal.

(a) Give the definition of T being strongly minimal.

(b) Show that all models of T are $\aleph_0$-homogeneous.

(c) Find an example to show that not all strongly minimal theories are κ-homogeneous for all κ. *Justify your answer.*

END OF PAPER