UNIVERSITY OF CAMBRIDGE

MAMA/139, NST3AS/139, MAAS/139

MAT3
MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 5:00pm

PAPER 139

NON-COMMUTATIVE NOETHERIAN RINGS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that k is a field.

(a) Define the notions of *left noetherian ring* and *right noetherian ring*.

Suppose that S is a ring that is generated by a subring R together with an element $x \in S$ such that $R + Rx = R + xR$. Show that if R is left noetherian then S is left noetherian.

(b) Recall that the first Weyl algebra $A_1(k)$ over k is defined by

$$A_1(k) := k\langle x, y\rangle/(yx - xy - 1).$$

Show that $A_1(k)$ is both left noetherian and right noetherian.

Show that $A_1(k)$ is simple if and only if $\mathrm{char}(k) = 0$.

(c) Consider the k-subalgebra T of $\mathrm{End}_k(k[t])$ generated by X and Y where $X(f) = tf$ for all $f \in k[t]$ and Y is the left inverse of X such that $Y(1) = 0$. Show that T is neither left noetherian nor right noetherian.

2

(a) State and prove Schur's lemma.

(b) Suppose that R is a ring and let V be a simple left R-module. Write $D = \mathrm{End}_R(V)^{\mathrm{op}}$ so that D acts naturally on V on the right. Suppose that X is a finite subset of V, and let

$$I := \bigcap_{x \in X} \mathrm{ann}_R(x).$$

Show that if $I \cdot y = \{0\}$ for some $y \in V$ then $y \in X \cdot D$, the D-linear span of X.

State and prove the Jacobson density theorem.

Deduce that if R is a primitive ring then there is a division ring D such that either:

(i) R is isomorphic to a matrix ring $\mathrm{Mat}_n(D)$; or

(ii) for every natural number n there is a subring R_n of R and a surjective ring homomorphism $R_n \to \mathrm{Mat}_n(D)$.

(c) Suppose that R is an arbitrary ring. By considering matrices e_{ij} with a 1 in entry (i, j) and 0's elsewhere, show that there is a 1-1 correspondence between simple R-modules and simple $\mathrm{Mat}_n(R)$-modules given by $S \mapsto S^n$.

Deduce that

$$J(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(J(R))$$

where $J(R)$ denotes the Jacobson radical of R.

3 Let R be a ring.

(a) Define an *injective (left) R-module*. State and prove Baer's criterion for injective left R-modules.

(b) Show that R is left noetherian if and only if every direct sum of injective left R-modules is injective.

(c) Suppose that R is a commutative principal ideal domain and E is an R-module. Show that E is injective if and only if E is divisible. Hence classify all indecomposable injective R-modules giving an explicit description of each one.

(d) Suppose that R is commutative and noetherian, J is an ideal of R and N is an essential submodule of a finitely generated R-module M. Show that if $JN = 0$ then there is $n \geqslant 1$ such that $J^n M = 0$. Deduce that the injective hull of R/J,

$$E(R/J) = \bigcup_{n \geqslant 1} \{x \in E(R/J) : J^n x = 0\}.$$

4 Suppose that R is a ring and k is a field of characteristic 0.

(a) Define a *left Ore set* in R. If S is a multiplicatively closed subset of R and M is a left R-module let

$$t_S(M) := \{m \in M : sm = 0 \text{ for some } s \in S\}.$$

Show that a multiplicatively closed subset S of R is a left Ore set if and only if for every left R-module M the set $t_S(M)$ is a submodule of M.

(b) Define a *prime ideal* of R. If P is a prime ideal in R let

$$\mathcal{C}_R(P) = \{r \in R : r + P \text{ is a regular element of } R/P\}.$$

Consider the ring

$$T_2(k) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in k \right\}$$

under usual matrix operations. What are the prime ideals of $T_2(k)$? For each prime ideal $P \lhd T_2(k)$ find the largest subset of $\mathcal{C}_{T_2(k)}(P)$ that is a left Ore set.

(c) Now suppose $\mathfrak{g} = kx \oplus ky$ is a k-Lie algebra with $[yx] = -x$. Show that for every $\lambda \in k$ there is a ring homomorphism $U(\mathfrak{g}) \to \mathrm{End}_k(k[t])$ such that

$$\theta_\lambda(y)(f) = t\partial_t(f) + \lambda f \text{ and } \theta_\lambda(x)(f) = \partial_t(f) \text{ for all } f \in k[t].$$

Describe all the $U(\mathfrak{g})$-submodules of $k[t]$ under the action induced by θ_λ.

Consider $I_\lambda := (x, y - \lambda) \lhd U(\mathfrak{g})$. Show that I_λ is a maximal ideal.

By considering the action of $U(\mathfrak{g})$ on $k \oplus kt$ via θ_λ, show that if $S \subset \mathcal{C}_{U(\mathfrak{g})}(I_\lambda)$ is a left Ore set then $S \subset \bigcap_{n \in \mathbb{N}_0} \mathcal{C}_{U(\mathfrak{g})}(I_{\lambda+n})$.

END OF PAPER