UNIVERSITY OF CAMBRIDGE

MAMA/136, NST3AS/136, MAAS/136

MAT3

MATHEMATICAL TRIPOS Part III

Friday 5 June 2026 9:00am to 11:00am

PAPER 136

LOCAL FIELDS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define an *absolute value* and a *non-archimedean absolute value* on a field K. Show that if $\mathrm{char}(K) > 0$ then all absolute values on K are non-achimedean.

(b) Let $(K, |\cdot|)$ be a non-archimedean valued field. Which of the following statements are always true, and which can be false. Justify your answers.

(i) If $x, y \in K$ then $\min(|x|, |y|) \leqslant |x + y| \leqslant \max(|x|, |y|)$.

(ii) If $x, y \in K$ with $|x| \neq |y|$ then $|x + y| = |x - y|$.

(iii) The ring $\{x \in K : |x| \leqslant 1\}$ is a Principal Ideal Domain (PID).

(iv) K is totally disconnected.

(c) State and prove a classification of the non-archimedean absolute values on $\mathbb{Q}$ (up to a suitable notion of equivalence, which you should define).

(d) Let $(K, |\cdot|)$ be a non-archimedean valued field. Show that if $\mathrm{char}(K) = 0$ and K is locally compact then K is a finite extension of $\mathbb{Q}_p$ for some prime p.

2 (a) State and prove a version of Hensel's lemma.

(b) Prove that if $m \in \mathbb{Z}_2$ then $1 + 8m \in (\mathbb{Q}_2^*)^2$. [*Hint: You may wish to consider the polynomial* $f(X) = X^2 - X - 2m$.] Prove that for each place v of $\mathbb{Q}$ we have

$$|\mathbb{Q}_v^*/(\mathbb{Q}_v^*)^2| = \frac{4}{|2|_v}.$$

(c) Let k be a field. Define the field of *Laurent power series* $k((t))$ and prove that it is complete with respect to a discrete valuation. If $K = \mathbb{F}_p((t))$ then for which primes p is the group $K^*/(K^*)^2$ finite? Justify your answer.

3 Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$ and uniformiser π. Let ζ_p be a primitive pth root of unity, and let $q = |\mathcal{O}_K/\pi\mathcal{O}_K|$.

(a) Show that for all r sufficiently large $(\pi^r\mathcal{O}_K, +) \cong (1 + \pi^r\mathcal{O}_K, \times)$. Deduce that $\mathcal{O}_K^*$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$.

(b) Show that if $a \in \mathcal{O}_K^*$ then the sequence $(a^{q^n})_{n\geqslant 0}$ converges in K to a $(q-1)$th root of unity which is congruent to $a \pmod{\pi}$.

(c) Determine the number of roots of unity in $\mathbb{Q}_p$ and in $\mathbb{Q}_p(\zeta_p)$ for all primes p.

4 (a) Let K be a p-adic field. Show that for each integer $n \geqslant 2$ there is a unique unramified extension L/K with $[L:K] = n$.

(b) Let L/K be a finite Galois extension of p-adic fields. Define the ramification subgroups of $G = \mathrm{Gal}(L/K)$. Show that $\cap_{n \geqslant 0} G_n = \{1\}$ and that there is an injective group homomorphism $G_0/G_1 \to k_L^*$ where k_L is the residue field of L.

(c) Let D_{2n} be the dihedral group of order $2n$.

(i) Find a Galois extension $L/\mathbb{Q}_2$ with $\mathrm{Gal}(L/\mathbb{Q}_2) \cong D_6$.

(ii) Clearly stating any further results about ramification groups that you need, show that there does not exist a Galois extension $L/\mathbb{Q}_2$ with $\mathrm{Gal}(L/\mathbb{Q}_2) \cong D_{10}$.

END OF PAPER