UNIVERSITY OF CAMBRIDGE

MAMA/133, NST3AS/133, MAAS/133

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 11 June 2026 2:00pm to 5:00pm

PAPER 133

GEOMETRIC GROUP THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be the wedge of two circles, $S^1 \vee S^1$, thought of as a graph with a single vertex x_0, so $\pi_1(X, x_0)$ is isomorphic to F_2, the free group of rank 2, with canonical generating set $\{a, b\}$.

(a) Draw pictures of the covering spaces of X corresponding to the following subgroups.

(i) The kernel of the homomorphism $F_2 \to \mathbb{Z}$ sending $a \mapsto 1$ and $b \mapsto 0$.

(ii) The kernel of the homomorphism $F_2 \to \mathbb{Z}$ sending $a \mapsto 2$ and $b \mapsto 3$.

(iii) The subgroup $\langle a \rangle \leqslant F_2$.

(b) Let Y be a connected graph, Z a connected subgraph and y_0 a vertex of Z. Consider a combinatorial loop α in Z, starting and ending at y_0. Explain why α is homotopic in Z (rel. endpoints) to a locally injective combinatorial loop β.

Let $p : \widetilde{Y} \to Y$ be the universal covering map. Prove that β lifts to an injective path in $\widetilde{Y}$, quoting any results that you need from lectures. Deduce that the inclusion map $Z \to Y$ induces an injective homomorphism $\pi_1(Z, y_0) \to \pi_1(Y, y_0)$.

(c) For a subgroup H of F_2, let (Y, y_0) be the corresponding based covering space of (X, x_0). Prove that, if H is finitely generated, then there is a finite subgraph $Z \subseteq Y$, containing y_0, such that the inclusion map induces an isomorphism $H \cong \pi_1(Z, y_0)$.

2 (a) Draw the Bass–Serre trees of the following amalgamated free products and HNN extensions.

(i) $\mathbb{Z} *_{2\mathbb{Z}=3\mathbb{Z}} \mathbb{Z}$

(ii) $\mathbb{Z} *_{\mathbb{Z} \sim 2\mathbb{Z}}$

(b) Consider the group $K = \langle a, b \mid aba^{-1}b \rangle$. Explain why K acts properly and cocompactly on the Euclidean plane. Prove that the subgroup generated by a^2 and b is isomorphic to $\mathbb{Z}^2$. What is its index in K?

(c) Consider the group

$$G = \langle a, c \mid ac^2a^{-1}c^2 \, , \, ca^2c^{-1}a^2 \rangle \, .$$

Write G as an amalgamated free product with edge group isomorphic to $\mathbb{Z}^2$, and draw the corresponding Bass–Serre tree.

(d) Prove that

$$a^{i_1}c^{j_1}a^{i_2} \dots c^{j_{n-1}}a^{i_n}c^{j_n} \neq 1$$

in G whenever all the exponents i_k and j_k are odd and $n \geqslant 1$.

3 Consider the following group G defined by a presentation.

$$G = \langle a, b, c, x, y \mid a^2x^{-1}, b^3y^{-1}, c^7xy, abc, [a, y], [b, x] \rangle .$$

(a) Compute the abelianisation of G. *[Hint: It may be useful to simplify the given presentation of G.]*

(b) Find a homomorphism from G onto a non-elementary Fuchsian group.

(c) Prove that x and y together generate the centre $Z(G)$.

Now suppose that G acts on a tree T by combinatorial isometries.

(d) Prove that, if x acts hyperbolically, then G preserves a line in T.

(e) Prove that, if $Z(G)$ acts trivially on T, then G acts trivially on T.

4 A *quasi-tree* is a geodesic metric space that is quasi-isometric to a tree.

(a) Use the Mostow–Morse lemma to show that any quasi-tree X is δ-hyperbolic, for some δ depending on X.

(b) A geodesic metric space X satisfies the *bottleneck property* if there is a constant C such that the following holds. Let $x, y \in X$, let $[x, y]$ be a geodesic and let m be the midpoint of $[x, y]$. For any continuous path $\alpha : [0, 1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$, there is $t \in [0, 1]$ such that $d(m, \alpha(t)) \leqslant C$. Prove that quasi-trees satisfy the bottleneck property.

(c) For some δ, give an example of a δ-hyperbolic metric space X that is not a quasi-tree. Justify your answer, using results from lectures.

(d) For *all* $\delta > 0$, give examples of δ-hyperbolic metric spaces X_δ that are not quasi-trees. Justify your answer.

END OF PAPER