MAMA/131, NST3AS/131, MAAS/131

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 2:00pm to 5:00pm

PAPER 131

RIEMANNIAN GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let (M, g) be a connected Riemannian manifold. Define the distance function $d(\cdot, \cdot)$ induced by g. State the Gauss lemma and deduce from it that $d(p, q) = 0$ holds for $p, q \in M$ if and only if $p = q$.

(b) Suppose that $\exp_p$ is defined on all of T_pM and $q \neq p$. Show that for each sufficiently small $\delta > 0$ there exists p_0 such that $d(p, p_0) = \delta$ and

$$d(p, p_0) + d(p_0, q) = d(p, q).$$

(c) What does it mean for g to be *geodesically complete*? State the Hopf–Rinow theorem.

(d) Let g and $\widehat{g}$ be two Riemannian metrics on M such that $\widehat{g}(X, X) \geqslant g(X, X)$ for all tangent vectors $X \in TM$. Show that if g is geodesically complete, then so is $\widehat{g}$.

2 (a) Let γ be a geodesic on a Riemannian manifold. State the formula for the second variation of the length of γ for the case of variation with fixed end-points, explaining the terms that appear.

(b) State and prove the Bonnet–Myers diameter theorem.

[*You may assume that for every two points p and q in a complete connected Riemannian manifold there exists a length-minimizing geodesic connecting p and q.*]

(c) Give an example to show that in the statement of the Bonner–Myers diameter theorem one cannot replace the hypothesis on Ricci curvature by a positive lower bound on the scalar curvature. [You should state standard results about curvature that you require but you do not need to prove them.]

3 (a) Define the *volume form* ω_g and the *Hodge $*$ operator* for an oriented Riemannian manifold, explaining briefly the auxiliary notions you require. Define the *Laplace–Beltrami operator.*

State the Hodge decomposition theorem.

(b) Is the volume form ω_g harmonic? Is ω_g parallel with respect to the Levi-Civita connection of g? Justify your answers.

(c) Let N be a 4-dimensional oriented Riemannian manifold. Show that every $\alpha \in \Omega^2(N)$ can be written as $\alpha = \alpha_+ + \alpha_-$ for some unique $\alpha_\pm$ satisfying $*\alpha_\pm = \pm\alpha_\pm$ (such $\alpha_\pm$ are called, respectively, the *self-dual* and *anti-self-dual* 2-forms). If N is also compact, show that every exact 3-form on N is the exterior derivative of a self-dual 2-form.

[*You may assume that $\delta = (-1)^{n(p+1)+1} * d*$ defines the formal L^2 adjoint of the exterior derivative d, but if you use other results about the Hodge $*$ or Laplace operator, then you should prove them.*]

4 (a) Let M be a complete Riemannian manifold. What is a *line* in M? Explain what is meant by M being *disconnected at infinity*. Show that if M is disconnected at infinity, then M must contain a line.

(b) State the Cheeger–Gromoll splitting theorem and the Hadamard–Cartan theorem.

(c) Let Y be a closed connected 3-dimensional manifold whose universal covering is not contractible. Show that $Y \times \mathbb{R}$ does not admit a complete Ricci-flat metric.

[*The Hopf–Rinow theorem may be assumed if accurately stated. You may assume that every 3-dimensional Ricci-flat manifold is flat.*]

END OF PAPER