UNIVERSITY OF CAMBRIDGE

MAMA/130, NST3AS/130, MAAS/130

MAT3

MATHEMATICAL TRIPOS Part III

Friday 12 June 2026 9:00am to 11:00am

PAPER 130

RAMSEY THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State and prove the canonical Ramsey theorem.

Deduce carefully the following statement: for any s there exists n such that whenever $[n]^{(2)}$ (the complete graph on vertex-set $[n]$) is coloured there exists a set $S \subset [n]$ of size s such that $S^{(2)}$ is coloured by one of the four canonical colourings.

(b) Suppose that the edges of the complete graph on $\mathbb{Q}$ are finitely coloured. Show that there must exist a set $A = \{x_1, x_2, \ldots\}$, where $x_1 < x_2 < \ldots$, that is monochromatic, and also a set $B = \{y_1, y_2, \ldots\}$, where $y_1 > y_2 > \ldots$, that is monochromatic.

Is it always possible to choose A and B in such a way that they have the same colour?

2 State and prove Rado's theorem for the case of a single equation. [You may assume van der Waerden's theorem.]

For fixed positive integers $a_1, \ldots, a_n$ and b, consider the equation $a_1x_1 + \ldots + a_nx_n = b$. Show that this equation is partition regular (meaning that whenever $\mathbb{N}$ is finitely coloured there is a monochromatic solution to the equation) if and only if b is a multiple of $a_1 + \ldots + a_n$.

3 (a) Show that there exists a non-principal ultrafilter on $\mathbb{N}$. Define the topological space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

State and prove Hindman's theorem. [You may assume that there is an idempotent for $+$ in $\beta\mathbb{N}$, as well as simple properties of ultrafilters, their quantifiers, and the operation $+$ on $\beta\mathbb{N}$.]

(b) Fix a non-principal ultrafilter $\mathcal{U}$. Show that, given a red-blue colouring of the edges of the complete graph on $\mathbb{N}$, we must have either $\forall_{\mathcal{U}}x\forall_{\mathcal{U}}y$ (xy is red) or $\forall_{\mathcal{U}}x\forall_{\mathcal{U}}y$ (xy is blue). Can these statements both be true?

Supposing that $\forall_{\mathcal{U}}x\forall_{\mathcal{U}}y$ (xy is red), prove that there is an infinite set $M \subset \mathbb{N}$ all of whose edges are red.

4 Prove that every regular m-gon is Ramsey. [If you use a result about A-invariant colourings of X^n then you must prove it.]

A finite subset X of $\mathbb{R}^d$, for some d, is called *edge Ramsey* if for every k there exists a finite set S in $\mathbb{R}^n$, for some n, such that whenever $S^{(2)}$, the complete graph on vertex-set S, is k-coloured there is a monochromatic set $Y \subset S$ that is isometric to X. (Here as usual Y being monochromatic means that all edges of Y have the same colour.) Which sets are edge Ramsey?

Now suppose that we weaken the condition that Y is isometric to X to merely the condition that Y is similar to X (meaning that we also allow enlargements – in other words, Y is similar to X if and only if Y is isometric to tX for some positive real t). Does this change the answer to the above question?

END OF PAPER