UNIVERSITY OF CAMBRIDGE

MAMA/128, NST3AS/128, MAAS/128

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 10 June 2026 9:00am to 11:00am

PAPER 128

FORCING AND THE CONTINUUM HYPOTHESIS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS	**SPECIAL REQUIREMENTS**
Cover sheet	None
Treasury tag	
Script paper	
Rough paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define

(i) the *definable power set* $\mathcal{D}(X)$,

(ii) the *constructible hierarchy* $\mathbf{L}_\alpha$, and

(iii) the *condensation sentence* (i.e., a sentence σ such that for any transitive set X, if $X \models \sigma$, then there is a limit ordinal λ such that $X = \mathbf{L}_\lambda$).

[*It is not necessary to give all details of the definition of the condensation sentence; instead, describe the general principles how to identify it.*]

(b) Let $T_\alpha := \{\varphi\,;\,\varphi$ is a sentence and $\mathbf{L}_\alpha \models \varphi\}$. Fix α and show that

(i) there are $\gamma > \beta > \alpha$ such that $T_\beta \neq T_\gamma$ and

(ii) there are $\gamma > \beta > \alpha$ such that $T_\beta = T_\gamma$.

2 Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a partial order.

(a) Define what it means that G is $\mathbb{P}$-*generic over* M.

As usual, we denote the set of functions from $\mathbb{N}$ to $\mathbb{N}$ by $\mathbb{N}^{\mathbb{N}}$ and the set of finite sequences of natural numbers by $\mathbb{N}^{<\mathbb{N}}$. For $s \in \mathbb{N}^{<\mathbb{N}}$, we write $|s|$ for its length. A function $f \in \mathbb{N}^{\mathbb{N}}$ is *eventually different over* M if for all $g \in \mathbb{N}^{\mathbb{N}} \cap M$, there is some n such that for all $k \geqslant n$, we have $g(k) \neq f(k)$. A function $f \in \mathbb{N}^{\mathbb{N}}$ is *infinitely equal over* M if for all $g \in \mathbb{N}^{\mathbb{N}} \cap M$, there are infinitely many k such that $g(k) = f(k)$. Consider the following partial orders:

(i) $\mathbb{Q}_0 := \mathbb{N}^{<\mathbb{N}}$, ordered by $s \leqslant t$ if t is an initial segment of s;

(ii) $\mathbb{Q}_1$ is the set of pairs (s, A) where $s \in \mathbb{N}^{<\mathbb{N}}$ and $A \subseteq \mathbb{N}^{\mathbb{N}}$ is a finite set, ordered by $(s, A) \leqslant (t, B)$ if t is an initial segment of s, $B \subseteq A$, and if $|t| \leqslant i < |s|$, then for all $f \in B$, $s(i) \neq f(i)$.

Let G_0 and G_1 be $\mathbb{Q}_0$-generic and $\mathbb{Q}_1$-generic over M, respectively, and

$$x_0(k) := n \text{ if there is an } s \in G_0 \text{ with } s(k) = n \text{ and}$$
$$x_1(k) := n \text{ if there is an } (s, A) \in G_1 \text{ with } s(k) = n.$$

(b) Determine whether x_0 or x_1 are eventually different or infinitely equal over M. Justify your (four) answers.

3

(a) (i) State *Hausdorff's formula* for cardinal exponentiation. [*You do not need to prove the formula.*]

(ii) Let X and Y be sets and κ be a cardinal. Define $\mathrm{Fn}(X, Y, \kappa)$.

[*In this question, you may assume without proof that* $\mathrm{Fn}(X, Y, \kappa)$ *is* κ*-closed* if κ is regular.]

Assume that M is a countable transitive model of ZFC with $\mathbb{P}, \kappa, \mu \in M$ such that $M \models$"$\mathbb{P}$ is a partial order and κ and μ are infinite cardinals".

(b) (i) State the relationship between $\mathbb{P}$ having the κ-chain condition in M and the preservation of cardinals in $\mathbb{P}$-generic extensions of M. Using this relationship, what preservations of cardinals can be derived from knowing that the cardinality of $\mathbb{P}$ is μ in M? [*You do not need to prove your claims.*]

(ii) State the relationship between $\mathbb{P}$ being $\aleph_1$-closed in M, the set $\wp(\mathbb{N})$ (i.e., the power set of $\mathbb{N}$) in $\mathbb{P}$-generic extensions of M, and the preservation of cardinals in $\mathbb{P}$-generic extensions of M. [*You do not need to prove your claim.*]

(c) Assume that $M \models 2^{\aleph_0} = \aleph_2$, let $\mathbb{P} = \mathrm{Fn}(\aleph_1^M \times \mathbb{N}, 2, \aleph_1^M)$ and G be $\mathbb{P}$-generic over M. Determine the value of $2^{\aleph_0}$ in $M[G]$. Justify your claim.

(d) Assume that $N \supseteq M$ is another countable transitive model of ZFC such that $\aleph_1^M$ is countable in N. In N consider $\mathbb{P} = \mathrm{Fn}(\aleph_1^N, 2, \aleph_1^N)$ and assume that $\mathbb{Q} := \{p \in \mathbb{P}\,;\, p$ is countable in $M\} \in N$. Let G and H be $\mathbb{P}$- and $\mathbb{Q}$-generic over N, respectively. Is $\wp(\mathbb{N}) \cap N[G] = \wp(\mathbb{N}) \cap N[H]$? Justify your claim.

END OF PAPER