UNIVERSITY OF CAMBRIDGE

MAMA/122, NST3AS/122, MAAS/122

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 9:00am to 12:00pm

PAPER 122

PROBABILISTIC COMBINATORICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that the following holds. If $G \sim G(n,p)$ and $p = (1-\varepsilon)/n$ then the largest connected component of G has at most $C_\varepsilon \log n$ vertices, with high probability.

(b) Again let $p = (1-\varepsilon)/n$. Show that every component of $G \sim G(n,p)$ of size at most $(1/3)\log n$ contains at most one cycle, with high probability.

(c) Let $p(n) \in (0,1)$ be a decreasing sequence with the property that if $G_n \sim G(n,p(n))$

$$\lim_{n\to\infty} \mathbb{P}\big(G_n \supset C_n\big) = 1.$$

Show that if $G_n \sim G(n,3p(n))$ then

$$\lim_{n\to\infty} \mathbb{P}\big(G_n \supset C_\ell \text{ for all } 3 \leqslant \ell \leqslant n\big) = 1.$$

2

(a) Show that

$$R(C_4, K_k) \geqslant c\left(\frac{k}{\log k}\right)^{4/3},$$

for some constant $c > 0$.

(b) State the "lopsided local lemma" and use it to prove

$$R(C_4, K_k) \geqslant \Omega(k^{4/3+c}),$$

for some $c > 0$. *[You may assume the lopsided local lemma.]*

(c) For $\ell \geqslant 4$, let H_ℓ be the graph which consists of a path on $\ell - 1$ vertices and another vertex which is joined to all the vertices of the path.

Show that for all ℓ and k sufficiently large (depending on ℓ), we have

$$R(H_\ell, K_k) \leqslant C\frac{\ell k^2}{\log k},$$

for some constant $C > 0$. *[You may assume results proved in the lectures, provided they are stated correctly.]*

3

(a) State Szemerédi's regularity lemma. Be sure to define the relevant terms first.

(b) Show that for every $\varepsilon > 0$ there exists $n(\varepsilon)$ so that if $n \geqslant n(\varepsilon)$ and $A \subset [n]$ with $|A| \geqslant \varepsilon n$ then there exist distinct $a, b, c, d, e \in A$ so that $a + b + c + d = 4e$.

(c) Say that a graph G on n vertices is (p, δ)-*dense* if $e(A, B) \geqslant p|A||B|$ for all $|A|, |B| \geqslant \delta n$.

Show that for every $\varepsilon > 0$ there exists $\delta > 0$ so that the following holds. If G is (p, δ)-dense with $p \geqslant \delta$ then G contains a subgraph G', with $V(G) = V(G')$ and such that

$$|e_{G'}(A, B) - p|A||B|| \leqslant \varepsilon n^2,$$

for all disjoint $A, B \subset V(G')$.

4

(a) For a graph H, define the Ramsey number $R(H)$. Show that if H has minimum degree d then

$$R(H) \geqslant 2^{d/2}.$$

(b) Show that if H is a bipartite graph on k vertices with maximum degree d then

$$R(H) = O(d^{2d}k).$$

(c) Let G be a bipartite graph with bipartition $A \cup B$ and $e(G) \geqslant p|A||B|$, where $p \in (1/3, 1]$. Show that for all $t \leqslant \min\{|A|, 100 \log |B|\}$ there exist *distinct* vertices $x_1, \ldots, x_t \in A$ with

$$|N(x_1) \cap \cdots \cap N(x_t)| \geqslant p^t|B|/2.$$

Now use this to show that $R(K_{t,t}) = O\big(t2^t\big)$.

(d) Show that $R(K_{k,k}) = O\big((\log k)2^k\big)$.

[Hint: It may be useful to first find $x_1, \ldots, x_t$, where $t = k - C \log k$, with $|N(x_1) \cap \cdots \cap N(x_t)| \geqslant c2^{-t}n$ in some colour.]

END OF PAPER