UNIVERSITY OF CAMBRIDGE

MAMA/120, NST3AS/120, MAAS/120

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 5:00pm

PAPER 120

LOGIC AND COMPUTABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Questions 1 and 2 carry 30 marks each. Question 3 carries 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define what is meant by a *Kripke model* for a propositional language $\mathcal{L}$.

(b) State the Completeness Theorem for the Kripke semantics of IPC. Use it to show that $(p \to q) \vee (q \to p)$ is not intuitionistically valid.

Fix a proposition ϕ. The *theory* of a world w relative to ϕ in a Kripke model $(S, \leqslant, \Vdash)$ is given by $\mathrm{Th}_\phi(w) = \{\psi \in \Phi \colon w \Vdash \psi\}$, where Φ is the set of subformulae of ϕ.

(c) Consider the equivalence relation $\sim$ on S given by $w \sim w'$ if $\mathrm{Th}_\phi(w) = \mathrm{Th}_\phi(w')$. Show that the quotient set $S/{\sim}$ can be made into a Kripke model that forces precisely the same propositions in Φ that S does.

A poset $(T, \leqslant)$ is *tree-like* if it has a minimum element t_0 and, for every element $t \in T$, the subset of predecessors $\{t' \in T \colon t' \leqslant t\}$ is linearly ordered by $\leqslant$.

(d) Let $(S, \leqslant_S, \Vdash_S)$ be a Kripke model with a $\leqslant_S$-least element s_0. Show that there is a Kripke model $(T, \leqslant_T, \Vdash_T)$ such that $(T, \leqslant_T)$ is tree-like and $S \Vdash_S \phi$ iff $T \Vdash_T \phi$ for all propositions ϕ.

(e) Suppose ϕ is a proposition with n subformulae. Show that if $S \Vdash \phi$ for all Kripke models $(S, \leqslant, \Vdash)$ with $|S| \leqslant n2^n$, then $\vdash_{\mathrm{IPC}} \phi$.

2

(a) Define the classes of Σ_1 and Π_1 formulas in the language of PA.

[*You do not need to define* Δ_0*-formulae.*]

(b) State the Diagonalisation Lemma for theories in the language of PA.

(c) State and prove the Crude Incompleteness Theorem.

[*You may assume that every total recursive function is* Σ_1*-represented in* PA^- *without proof.*]

(d) Let ϕ be a sentence in the language of PA such that $\mathbb{N} \models \phi$. Show that there is a countable model of PA^- that satisfies ϕ, but that does not satisfy the same first-order $\mathcal{L}_{\mathrm{PA}}$-sentences as $\mathbb{N}$.

[*You may assume the Gödel-Rosser Theorem without proof.*]

(e) What does it mean for subsets $A, B \subseteq \mathbb{N}$ to be *recursively inseparable*?

(f) Fix a Gödel numbering $\ulcorner - \urcorner$ of sentences in the language of PA. Show that the sets $A = \{\ulcorner \sigma \urcorner \colon \mathrm{PA}^- \vdash \sigma\}$ and $B = \{\ulcorner \sigma \urcorner \colon \mathrm{PA}^- \vdash \neg\sigma\}$ are recursively inseparable.

3

[*Throughout this question, we fix a type variable* σ *for the simply-typed* λ*-calculus* $\lambda_{\Pi}(\to)$.]

(a) Define what it means for a partial function $\mathbb{N}^k \to \mathbb{N}$ to be λ-*definable.*

(b) State the Strong Normalisation Theorem for the simply-typed λ-calculus. Give an example of an untyped λ-term that is not strongly normalising.

In what follows, view each Church numeral $c_n = \lambda f\colon \sigma \to \sigma.\ \lambda s\colon \sigma.\ f^{(n)}(s)$ as a term of type $\mathrm{Nat} = (\sigma \to \sigma) \to (\sigma \to \sigma)$ in the simply typed λ-calculus.

(c) Give an example of partial function $\mathbb{N} \to \mathbb{N}$ that is λ-definable by a term F, but such that $\not\Vdash F\colon \mathrm{Nat} \to \mathrm{Nat}$. Justify your claim.

[*You may assume the Strong Normalisation Theorem without proof.*]

(d) Show that there are exactly two closed simply typed λ-terms $\top$ and $\bot$ (up to $\beta\eta$-equivalence) such that $\Vdash \top, \bot\colon \mathrm{Bool}_\sigma$, where Bool_σ is the type $\sigma \to (\sigma \to \sigma)$.

(e) Provide a simply-typed λ-term OR such that:

(i) $\Vdash \mathrm{OR}\colon \mathrm{Bool}_\sigma \to (\mathrm{Bool}_\sigma \to \mathrm{Bool}_\sigma)$;

(ii) For all λ-terms a and b where $\Vdash a, b\colon \mathrm{Bool}_\sigma$, we have $\mathrm{OR}\ a\ b \equiv_\beta \top$ if $a \equiv_\beta \top$ or $b \equiv_\beta \top$, and $\mathrm{OR}\ a\ b \equiv_\beta \bot$ otherwise.

Consider a *finite binary transition system*, i.e., a pair (Q, δ) where $Q = \{q_0, \cdots, q_{n-1}\}$ is a finite set and $\delta\colon Q \times \{0,1\} \to Q$ is a function. Recursively define

(i) $\delta^*(q, \epsilon) = q$

(ii) $\delta^*(q, wa) = \delta(\delta^*(q, w), a))$,

for each $q \in Q$, where ϵ is the empty string and wa is the result of appending the symbol $a \in \{0,1\}$ to the finite string $w \in \{0,1\}^*$.

(f) Suppose there are exactly n distinct closed λ-terms $\underline{q_0}, \cdots, \underline{q_{n-1}}$ in $\beta\eta$-normal form with $\Vdash \underline{q_i}\colon \sigma$ for all $i < n$. For each string $w \in \{0,1\}^*$, construct a simply-typed λ-term $\underline{w}$ such that $\Vdash \underline{w}\colon \sigma \to \sigma$ and $\underline{w}\ \underline{q_0} \equiv_\beta \underline{\delta^*(q_0, w)}$.

[*You may assume that there is a term* $\mathrm{eq}_\sigma\colon \sigma \to (\sigma \to \mathit{Bool}_\sigma)$ such that $\mathrm{eq}_\sigma\ x\ \underline{q_i} \equiv_\beta \top$ if $x \equiv_{\beta\eta} q_i$ and $\mathrm{eq}_\sigma\ x\ \underline{q_i} \equiv_\beta \bot$ otherwise, for all $i < n$.]

(g) Let $F \subseteq Q$. Using item (f), explain why $L = \{w \in \{0,1\}^*\colon \delta^*(q_0, w) \in F\}$ is recursive.

[*For this item, you may assume results from the lectures, provided that you state them precisely and correctly.*]

END OF PAPER