UNIVERSITY OF CAMBRIDGE

MAMA/119, NST3AS/119, MAAS/119

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 9:00am to 12:00pm

PAPER 119

CATEGORY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by the statement that a category is *well-powered.* Show that if $\mathcal{C}$ is a small category then the functor category $[\mathcal{C}, \mathbf{Set}]$ is both well-powered and well-copowered. [You may assume that all monomorphisms and epimorphisms in this category are pointwise monic and epic, and also that every morphism factors, uniquely up to isomorphism, as an epimorphism followed by a monomorphism.]

Show that specifying a terminal object of a category $\mathcal{C}$ is equivalent to specifying a colimit for the identity functor $\mathcal{C} \to \mathcal{C}$, considered as a diagram of shape $\mathcal{C}$ in $\mathcal{C}$.

A *local state classifier* for a category $\mathcal{C}$ is a colimit for the inclusion functor $\mathcal{C}_m \to \mathcal{C}$, where $\mathcal{C}_m$ is the non-full subcategory of all objects of $\mathcal{C}$ and monomorphisms between them. Show that $[\mathcal{C}, \mathbf{Set}]$ has a local state classifier for any small category $\mathcal{C}$. [*Hint*: consider a functor whose value at an object A is the set of (isomorphism classes of) quotients of the functor $\mathcal{C}(A, -)$.] Show also that the functor category $[\mathbb{Z}, \mathbf{Set}_f]$, where $\mathbb{Z}$ is the additive group of integers and $\mathbf{Set}_f$ is the category of finite sets, does not have a local state classifier. [*Hint*: show that, for each positive integer n, there is a cone under the inclusion functor whose apex A_n has n elements, and whose legs are collectively surjective.]

2 Explain what is meant by an *idempotent* morphism in a category, and by the assertion that an idempotent is *split.* Show that, given a class $\mathcal{E}$ of idempotents in a category $\mathcal{C}$ which includes all identity morphisms, there is a full and faithful functor $\mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$ which maps all members of $\mathcal{E}$ to split idempotents, and such that any functor $\mathcal{C} \to \mathcal{D}$ sending all members of $\mathcal{E}$ to split idempotents factors through $\mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$.

Show that any preorder on a set is an idempotent in the category $\mathbf{Rel}$ of sets and relations, but that it is split only if it is an equivalence relation.

An element a of a complete lattice A is called *join-inaccessible* if, whenever $a \leqslant \bigvee S$ for some $S \subseteq A$, there exists $s \in S$ with $a \leqslant s$; A is called *completely algebraic* if every $a \in A$ is a join of join-inaccessible elements. Show that A is completely algebraic iff there is a power-set PB and order-preserving maps $i : A \to PB$ and $r : PB \to A$ such that $ri = 1_A$, $(r \dashv i)$ and i preserves all joins. [*Hint for 'only if'*: take B to be the set of join-inaccessible elements of A.] Hence show that if $\mathcal{E}$ is the class of preorders in $\mathbf{Rel}$, then $\mathbf{Rel}[\check{\mathcal{E}}]$ is equivalent to the full subcategory of $\mathbf{CSLat}$, the category of complete lattices and join-preserving maps, whose objects are the completely algebraic lattices. [You may assume the existence of a full and faithful poset-enriched functor $\mathbf{Rel} \to \mathbf{CSLat}$ defined on objects by the power-set construction, and the fact that all idempotents in $\mathbf{CSLat}$ are split.]

3 Define the category $(B \downarrow F)$, where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and B is an object of $\mathcal{D}$. Show that F has a left adjoint iff, for all $B \in \text{ob}\,\mathcal{D}$, the category $(B \downarrow F)$ has an initial object.

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *final* if, for any object B of $\mathcal{D}$, the category $(B \downarrow F)$ is connected. Show that if F is final and $D\colon \mathcal{D} \to \mathcal{E}$ is an arbitrary functor, then any cone under DF extends uniquely to a cone under D, and deduce that if $\mathcal{E}$ has colimits of shape $\mathcal{C}$ then it also has colimits of shape $\mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a *discrete fibration* if, given any morphism $f : B \to FA$ in $\mathcal{D}$, there exists a unique $\widetilde{f}\colon \widetilde{B} \to A$ in $\mathcal{C}$ with $F\widetilde{f} = f$. Show that any functor $F : \mathcal{C} \to \mathcal{D}$ can be factored as the composite of a final functor $G : \mathcal{C} \to \mathcal{E}$ and a discrete fibration $H : \mathcal{E} \to \mathcal{D}$. [*Hint*: take the objects of $\mathcal{E}$ to be pairs (B, C) where $B \in \text{ob}\,\mathcal{D}$ and C is a connected component of $(B \downarrow F)$.]

4 Explain briefly what is meant by the assertion that an adjunction is *monadic*. State, without proof, a lemma giving a criterion for the Eilenberg–Moore comparison functor for a given adjunction to have a left adjoint, and deduce the Precise Monadicity Theorem.

For each natural number n, let $\mathcal{C}_n$ be the category whose objects are sets A equipped with n partial unary operations $\alpha_1, \alpha_2, \ldots, \alpha_n$, subject to the conditions that $\alpha_1(a)$ is defined for all a, and for $m > 1$ $\alpha_m(a)$ is defined iff ($\alpha_{m-1}(a)$ is defined and) $\alpha_{m-1}(a) = a$. Show that the forgetful functor $\mathcal{C}_{n+1} \to \mathcal{C}_n$ has a left adjoint F_n, and that the adjunction is monadic. [*Hint*: we may take $F_n(A)$ to have underlying set $A \times \{0\} \cup A_n \times \mathbb{N}$, where A_n is the set of fixed points of α_n, and for $a \in A_n$ we have $\alpha_{n+1}(a, 0) = (a, 1)$ and $\alpha_1(a, m) = (a, m + 1)$ for all $m > 0$.]

Is the composite adjunction $\mathcal{C}_n \leftrightarrows \mathcal{C}_{n+1} \leftrightarrows \mathcal{C}_{n+2}$ monadic? Justify your answer.

5 Define the notion of *Lawvere theory*, and of a *model* for such a theory in a category with finite products. Explain how a finitary monad $\mathbb{T}$ on **Set** (that is, one whose underlying functor preserves filtered colimits) gives rise to a Lawvere theory $\mathcal{T}$. [You may assume the result that $\mathbf{Set}^{\mathbb{T}}$ is equivalent to the category of $\mathcal{T}$-models in **Set**.]

Now let $\mathbb{T}$ be a finitary monad such that $\mathbf{Set}^{\mathbb{T}}$ is additive. Show that the corresponding Lawvere theory $\mathcal{T}$ is also additive, and deduce that any $\mathcal{T}$-model in a category $\mathcal{C}$ has the structure of an internal abelian group in $\mathcal{C}$. Show also that every n-ary operation α of the theory (that is, every morphism $[1] \to [n]$ in $\mathcal{T}$) satisfies a unique equation of the form

$$\alpha(x_1, x_2, \ldots, x_n) = \alpha_1(x_1) + \alpha_2(x_2) + \cdots + \alpha_n(x_n)$$

where the α_i are unary operations. Hence show that $\mathbf{Set}^{\mathbb{T}}$ is equivalent to the category of R-modules, where R is the ring of endomorphisms of $[1]$ in $\mathcal{T}$.

6 Define the notion of *abelian category*, and show that in an abelian category every monomorphism is the kernel of its own cokernel. Deduce that an abelian category is well-powered iff it is well-copowered.

Explain what is meant by the category $\mathrm{c}\mathcal{A}$ of *complexes* in an abelian category $\mathcal{A}$, and by the *homology objects* of a complex. Prove that the definition of homology objects is self-dual.

Show that, for any integer n, the functor $\mathcal{A} \to \mathrm{c}\mathcal{A}$ which sends an object A to the complex $A[n]$ with A in dimension n and 0 in all other dimensions, has left and right adjoints. Deduce the Mayer–Vietoris Theorem which says that a short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ in $\mathrm{c}\mathcal{A}$ gives rise to a long exact sequence of homology objects in $\mathcal{A}$. [You may assume the Snake Lemma provided you state it precisely.]

END OF PAPER