UNIVERSITY OF CAMBRIDGE

MAMA/118, NST3AS/118, MAAS/118

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 10 June 2026 2:00pm to 5:00pm

PAPER 118

COMPLEX MANIFOLDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose X is a closed subset of $\mathbb{CP}^2$ defined by the vanishing locus of a non-zero homogeneous polynomial $F \in \mathbb{C}[Z_0, Z_1, Z_2]$ of degree $d \geqslant 2$.

(a) Show that if grad $F(p) \neq 0$ whenever $F(p) = 0$ for $p \neq 0$, then X is *complex submanifold* of $\mathbb{CP}^2$ with complex dimension 1.

(b) State the *adjunction formula* for X and determine the first Chern class $c_1(T_X)$ as a class in $H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{R})$.

(c) Let ω_{FS} be the Fubini-Study form on $\mathbb{CP}^2$. Define what is meant by a *Kähler form* and show that the restriction $\omega_X := \omega_{FS}|_X$ gives a Kähler form on X.

(d) Let ∇ be the Chern connection on T_X of the Hermitian metric associated to ω_X. Show that $iF_\nabla = \lambda\omega_X$ for some smooth function $\lambda \in C^\infty(X;\mathbb{R})$ and determine

$$\frac{1}{2\pi}\int_X \lambda \, dvol_g$$

in terms of the degree d of F.

(e) What is λ in terms of the metric g associated to the Kähler form ω_X on X? [There is no need to prove your answer].

2 Suppose $\Lambda \subset \mathbb{C}^n$ is a discrete additive subgroup of $\mathbb{C}^n$ that is isomorphic to $\mathbb{Z}^{2n}$ (called a *lattice*). Consider Λ acting on $\mathbb{C}^n$ by translation, that is, we define the group action by $(\lambda, z) \mapsto z + \lambda$ for $\lambda \in \Lambda$. Call the quotient space X and the projection map $\pi : \mathbb{C}^n \to X$. We call X a *complex torus*.

(a) Show that X has the structure of a compact complex manifold, with a natural Kähler form ω_X so that $\pi^*\omega_X = \omega_0$, the standard Kähler form on $\mathbb{C}^n$.

Suppose $\alpha \in C^\infty(X, \wedge^{p,q}T^*X)$ is given by

$$\alpha = \sum_{I,J} \alpha_{I,J}(z, \bar{z}) dz_I \wedge d\bar{z}_J$$

where the sum is taken over all index sets $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ with $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and $1 \leqslant j_1 < \cdots < j_q \leqslant n$.

(b) Show that $\Delta\alpha = 0$ if and only if every $\alpha_{I,J}$ is constant.

(c) State the *Hodge decomposition theorem* for the de Rham cohomology of a compact Kähler manifold and determine $h^{p,q}(X)$ for a complex torus X.

Recall if α is a closed 1-form representing a class in $H^1(X; \mathbb{C})$ and γ a smooth curve representing a class in $H_1(X; \mathbb{Z})$, then the cap product $\frown: H_1(X; \mathbb{Z}) \otimes H^1(X; \mathbb{C}) \to \mathbb{C}$ is given by

$$[\gamma] \frown [\alpha] = \int_\gamma \alpha.$$

(d) Use this to prove that there exists a pair of diffeomorphic but not biholomorphic complex manifolds of dimension n (for every $n \geqslant 1$).

3 Suppose that X is a smooth manifold with almost complex structure J.

(a) Define the complex vector bundles $\wedge^{p,q}T^*X$ and the differential operator $\bar{\partial} : C^\infty(X, \wedge^{p,q}T^*X) \to C^\infty(X, \wedge^{p,q+1}T^*X)$.

(b) Prove that J is integrable if and only if $\bar{\partial}^2 f = 0$ for all smooth functions $f : X \to \mathbb{C}$.

Now suppose X is a complex manifold (i.e. J is integrable) and that E is a *smooth* complex line bundle. We say a complex-linear map $\bar{\partial}_E : C^\infty(X, E) \to C^\infty(X, \wedge^{0,1}T^*X \otimes E)$ is a *partial connection* if it satisfies the Leibniz rule

$$\bar{\partial}_E(fs) = (\bar{\partial} f) \otimes s + f\bar{\partial}_E s$$

for any smooth section s of E and smooth function $f : X \to \mathbb{C}$.

(c) Explain how to extend the partial connection $\bar{\partial}_E$ to a complex-linear map

$$\bar{\partial}_E : C^\infty(X, \wedge^{p,q}T^*X \otimes E) \to C^\infty(X, \wedge^{p,q+1}T^*X \otimes E)$$

satisfying a graded Leibniz rule, and prove that the square $\bar{\partial}_E^2$ is linear over $C^\infty(X, \mathbb{C})$.

(d) Show that if E is a *holomorphic* line bundle over X then there exists a well-defined partial connection $\bar{\partial}_E$ so that $\bar{\partial}_E^2 = 0$.

4 Let X be a compact complex manifold, and suppose $Z \subset X$ is a codimension-r (smooth closed) complex submanifold.

(a) Define the *Dolbeault cohomology* of X and explain why Z defines a class $[Z]$ in $H^{r,r}(X) \cap H^{2r}(X;\mathbb{Z}) \subset H^{2r}(X;\mathbb{C})$.

(b) Suppose D_1, D_2 are two smooth complex hypersurfaces in X that are linearly equivalent. Show that there exists a holomorphic map $p : X \to \mathbb{CP}^1$ so that $D_1 = p^{-1}(0)$ and $D_2 = p^{-1}(\infty)$. Conclude that $[D_1] = [D_2]$ in Dolbeault cohomology.

(c) Suppose $\pi : \tilde{X} \to \mathbb{CP}^2$ is the blowup of $\mathbb{CP}^2$ at a point $p \in \mathbb{CP}^2$. Show that $\tilde{X}$ is independent of the choice of point p, up to biholomorphism. Moreover, show that if D_1 and D_2 are two smooth irreducible hypersurfaces that are linearly equivalent in $\mathbb{CP}^2$, then $\pi^{-1}(D_1)$ and $\pi^{-1}(D_2)$ are linearly equivalent in $\tilde{X}$.

(d) Determine the dimensions $h^{p,q}(\tilde{X})$ of the Dolbeault cohomology groups $H^{p,q}(\tilde{X})$. *[Hint: $b_2(\tilde{X}) = 2$.]*

END OF PAPER