UNIVERSITY OF CAMBRIDGE

MAMA/115, NST3AS/115, MAAS/115

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 2:00pm to 5:00pm

PAPER 115

DIFFERENTIAL GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let M be an n-dimensional embedded submanifold inside $\mathbb{R}^{n+m}$. (a) State the definition of the second fundamental form. Explain why the second fundamental form $A(X,Y)$ is symmetric. (b) State without proof the Gauss-Codazzi equation. (c) Use this to prove that the sectional curvature of the unit sphere S^n with the round metric is equal to one. Compute the scalar curvature of S^n.

2 Let (M,g) be a compact oriented Riemannian manifold, and let $f : M \to \mathbb{R}$ be a smooth function. (a) Describe how the metric g induces the volume form on M. For a smooth function $u : M \to \mathbb{R}$, we define the energy functional

$$E(u) = \frac{1}{2}\int_M |\nabla u|_g^2 dvol - \int_M fudvol.$$

Write out the energy functional in local coordinate notation. (b) Suppose u is a minimizer of the functional, derive the Euler-Lagrange equation, written in local coordinates. (c) When the minimizer exists, deduce $\int_M fdvol = 0$.

3 Let $G = Sp(2n,\mathbb{R}) = \{A \in Mat_{2n\times 2n}(\mathbb{R}) : A^T\Omega A = \Omega\}$, where $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
(a) Check that G is a group. Prove that the group G is an embedded submanifold of the vector space $Mat_{2n\times 2n}(\mathbb{R})$ of $2n \times 2n$ real matrices; state carefully the result you use. (b) Compute the dimension of G. (c) Prove that the multiplication map $G \times G \to G$ is a smooth map.

4 (a) State the definition of minimal surface. State without proof the first variation formula for the area of an n-dimensional embedded submanifold M inside the Euclidean space $\mathbb{R}^{n+m}$; be precise about your sign convention.

(b) Given any smooth function $f : \mathbb{R}^{n+m} \to \mathbb{R}$, state and prove the formula to compute the Laplacian on a submanifold M, given the information on the submanifold and the ambient derivatives of f.

(c) Prove that there is no compact minimal surface inside $\mathbb{R}^{n+m}$, using the maximum principle.

END OF PAPER