UNIVERSITY OF CAMBRIDGE

MAMA/114, NST3AS/114, MAAS/114

MAT3

MATHEMATICAL TRIPOS Part III

Friday 5 June 2026 2:00pm to 5:00pm

PAPER 114

ALGEBRAIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $C_\bullet$ be a chain complex consisting of free abelian groups and $n \in \mathbb{Z} \setminus \{0\}$. Construct long exact sequences of the form

$$\cdots \longrightarrow H_i(C_\bullet) \longrightarrow H_i(C_\bullet) \longrightarrow H_i(C_\bullet \otimes \mathbb{Z}/n) \xrightarrow{\hat{\beta}} H_{i-1}(C_\bullet) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H_i(C_\bullet \otimes \mathbb{Z}/n) \longrightarrow H_i(C_\bullet \otimes \mathbb{Z}/n^2) \longrightarrow H_i(C_\bullet \otimes \mathbb{Z}/n) \xrightarrow{\beta} H_{i-1}(C_\bullet \otimes \mathbb{Z}/n) \longrightarrow \cdots$$

describing precisely all the maps involved. *[You may use any results from lectures providing that they are clearly stated.]*

Explain why there is a map relating the two long exact sequences you constructed, and hence prove that $\beta^2 = 0$. Let $\beta H_*(C_\bullet; n)$ denote the homology of the chain complex given by the sequence of abelian groups $H_*(C_\bullet \otimes \mathbb{Z}/n)$ with differential β.

Compute $\beta H_*(C_*(\mathbb{RP}^3); 2)$. *[You may use any facts about the homology of $\mathbb{RP}^3$ with any coefficients as long as they are clearly stated.]*

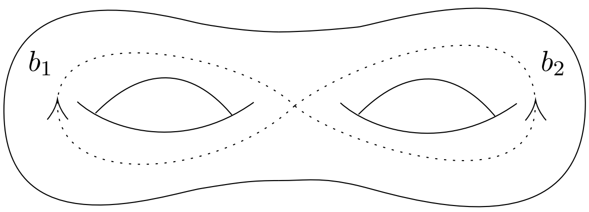
Suppose that $C_\bullet$ consists of finitely-generated free abelian groups, and p is a prime number. By explaining how to decompose $C_\bullet$ as a direct sum of simple chain complexes, show that:

(i) a $\mathbb{Z}$-summand of $H_i(C_\bullet)$ contributes a $\mathbb{Z}/p$-summand to $\beta H_i(C_\bullet; p)$,

(ii) a $\mathbb{Z}/m$-summand of $H_i(C_\bullet)$ contributes a $\mathbb{Z}/p$-summand to both $\beta H_i(C_\bullet; p)$ and $\beta H_{i+1}(C_\bullet; p)$ if $p^2 | m$, and contributes nothing if $p^2 \nmid m$.

2 Describe the calculation of the cohomology groups and cup products of the oriented surface Σ_g of genus g. *[You may use any results from the course, including the calculation of the (co)homology of spheres.]*

Consider the space X formed by attaching a 2-cell to Σ_2 along the loop which goes along b_1 then along b_2, then backwards along b_1 and then backwards along b_2, in the following picture:



Determine the cohomology ring of X.

3 Let R be a commutative ring. What does it mean to say that a d-dimensional vector bundle $\pi : E \to B$ is R-oriented? State the Thom isomorphism theorem, and describe the Euler class and Gysin sequence. *[You do not need to derive the Gysin sequence.]*

Calculate the ring $H^*(\mathbb{CP}^n; \mathbb{Z}/2)$, and write down the ring $H^*(\mathbb{RP}^m; \mathbb{Z}/2)$.

Determine the $\mathbb{Z}/2$-Euler class of the 2-dimensional real vector bundle

$$\gamma^{\mathbb{C}}_{1,n+1} \otimes_{\mathbb{R}} \gamma^{\mathbb{R}}_{1,m+1} \longrightarrow \mathbb{CP}^n \times \mathbb{RP}^m.$$

[You may use the Künneth theorem, as well as the identity $e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$.] When $(n, m) = (2, 2)$, use this to compute the $\mathbb{Z}/2$-cohomology groups of the unit sphere bundle $X := S(\gamma^{\mathbb{C}}_{1,3} \otimes_{\mathbb{R}} \gamma^{\mathbb{R}}_{1,3})$.

4 State the Lefschetz fixed point theorem.

Let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map which is $\mathbb{Z}^2$-periodic in the sense that it satisfies:

1. $\tilde{f}(0) = 0$,
2. $\tilde{f}(x + v) - \tilde{f}(x) \in \mathbb{Z}^2$ for all $x \in \mathbb{R}^2$ and all $v \in \mathbb{Z}^2$.

Let $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ denote the induced map.

Show that the function $A := \tilde{f}|_{\mathbb{Z}^2} : \mathbb{Z}^2 \to \mathbb{Z}^2$ agrees with

$$f_* : H_1(\mathbb{R}^2/\mathbb{Z}^2; \mathbb{Z}) \longrightarrow H_1(\mathbb{R}^2/\mathbb{Z}^2; \mathbb{Z}),$$

under an identification of the abelian group $H_1(\mathbb{R}^2/\mathbb{Z}^2; \mathbb{Z})$ with $\mathbb{Z}^2$ which you should describe, and hence deduce that A is a homomorphism.

Now suppose in addition that $\mu > 1$ is such that

3. $|\tilde{f}(x) - \tilde{f}(y)| \geqslant \mu \cdot |x - y|$ for all $x, y \in \mathbb{R}^2$.

Show that f has at least $(\mu - 1)^2$ fixed points.

[You may use the following fact from linear algebra: if a real matrix B satisfies $|Bv| \geqslant \lambda |v|$ for all real vectors v, then every eigenvalue λ_i of B satisfies $|\lambda_i| \geqslant \lambda$.]

END OF PAPER