MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 9 June 2026 9:00am to 12:00pm

PAPER 113

ALGEBRAIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $f : X \to Y$ be a morphism of Noetherian schemes. State the valuative criteria for separatedness and properness for the morphism f.

(b) Let $f : X \to Y$, $g : Y \to Z$ be morphisms of Noetherian schemes with f of finite type. Suppose that $g \circ f$ is proper and g is separated. Show that f is proper.

(c) Assume given the fact that the morphism $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is separated. Show that f satisfies the valuative criterion for properness whenever the valuation ring is a discrete valuation ring. [*Hint: make use of the standard affine open cover of* $\mathbb{P}^n_{\mathbb{Z}}$, *and make use of the discrete valuation* $\nu : K^* \to \mathbb{Z}$ *where* K *is the field of fractions of a discrete valuation ring.*]

2 In the following, let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism of ringed spaces. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules and $\mathcal{E}$ a sheaf of $\mathcal{O}_Y$-modules.

(a) Define the sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
Define the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{F}$.
Define the sheaf of $\mathcal{O}_X$-modules $f^*\mathcal{E}$.

(b) Define what it means for $\mathcal{E}$ to be a locally free $\mathcal{O}_Y$-module of finite rank.

(c) Construct a morphism

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} \to f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}).$$

Prove that the morphism you construct has the property that if $\mathcal{E}$ is locally free of finite rank, then the morphism is an isomorphism.

3 (a) Let X be a topological space, $\mathcal{U}$ an open covering of X, and $\mathcal{F}$ a sheaf on X. Define the *Čech cohomology of $\mathcal{F}$ with respect to the cover $\mathcal{U}$*, $\check{H}^p(\mathcal{U}, \mathcal{F})$.

(b) Let k be a field. Let $X = \operatorname{Spec} k[x, y]$ and $U = X \setminus \{(x, y)\}$ (removing the maximal ideal corresponding to the origin). Let $V = U \setminus V(f)$ for $f \in k[x, y]$ a polynomial with $f(0, 0) \neq 0$. Calculate $H^0(V, \mathcal{O}_V)$ and show that $H^1(V, \mathcal{O}_V)$ is infinite-dimensional.

(c) Let X and U be as in (b). Let Y be the scheme obtained by gluing two copies of X together via the identity on U: this is the affine plane with doubled origin. Denote by $X_1, X_2 \subset Y$ the two copies of X viewed as open subsets of Y. Let $\mathcal{U} = \{X_1, X_2\}$ be the open cover of Y. Show that $\check{H}^p(\mathcal{U}, \mathcal{O}_Y) \neq H^p(Y, \mathcal{O}_Y)$ for some p. Why doesn't this contradict a result from lecture?

[*Hint: You may use the* Mayer-Vietoris sequence *without proof. This says that given a topological space $X = U \cup V$ with $U, V \subseteq X$ open, and $\mathcal{F}$ a sheaf on X, there is a long exact sequence*

$$\begin{aligned} 0 \to & H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \\ \to & H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}) \oplus H^1(V, \mathcal{F}) \to H^1(U \cap V, \mathcal{F}) \to \cdots] \end{aligned}$$

4 (a) Define what it means for a morphism of schemes $f : X \to Y$ to be a *closed immersion*. Define what it means for X to be a *closed subscheme* of Y.

(b) Let X be a scheme, and $Z \subseteq X$ a closed subset. Explain how to give Z the structure of a closed subscheme of X such that the scheme structure on Z is reduced. Show that given any closed immersion $Z' \to X$ inducing a homeomorphism between Z' and Z, there is a unique factorization $Z \to Z' \to X$ of the given closed immersion $Z \to X$. [*Hint: You may use the fact that any closed subscheme of* $\operatorname{Spec} A$ *is isomorphic to* $\operatorname{Spec} A/I$ *for some ideal I.*]

(c) Let $f : X \to Y$ be a morphism. The *scheme-theoretic image of f* is a closed subscheme Z of Y with the following property: the morphism f factors through Z, and if Z' is any other closed subscheme of Y through which f factors, then $Z \to Y$ also factors as $Z \to Z' \to Y$.

If X and Y are affine schemes, construct the scheme-theoretic image of f. Show in this case that the set underlying Z is the closure of $f(X)$.

END OF PAPER