UNIVERSITY OF CAMBRIDGE

MAMA/112, NST3AS/112, MAAS/112

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 2:00pm to 5:00pm

PAPER 112

KNOTS AND KNOT CONCORDANCES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 We say that an oriented knot K is *composite* if $K = K_1 \sharp K_2$ for some oriented knots K_1 and K_2 that are not unknots, and *prime* if it is not composite. You can use the fact that for any embedded 2-sphere Σ in S^3, its complement $S^3 \setminus \Sigma$ consists of two 3-balls.

(a) Define Seifert surfaces of a knot K and describe how its Seifert form is defined. Also, describe how the Levine–Tristram signature of K is computed from its Seifert form. Show that, if K satisfies

$$\{z \in \mathbb{C} \mid \Delta_K(z) = 0,\ |z| = 1\} = \emptyset,$$

then the Levine–Tristram signature of K vanishes identically.

(b) Let P be a pattern with winding number 0. Show that there exists an integer $g_P \geqslant 0$, depending only on P, such that for any knot K, we have $g_s(P(K)) \leqslant g_P$.

(c) For any oriented knots K and K', show that $g_s(K \sharp K') = g_s(K) + g_s(K')$. Deduce that the torus knot $T_{2,3}$ is prime. You are not allowed to cite the corresponding theorem in the lectures.

(d) Show that any knot is a connected sum of finitely many prime knots.

(e) Given a knot K, we say that an embedded 2-sphere $\Sigma \subset S^3$ is a *splitting sphere of* K if it intersects K transversely at two points. A splitting sphere Σ of K is *trivial* if there exists a ball B bounded by Σ and an embedded disk $D \subset B$ such that $\mathrm{int}(D) \subset \mathrm{int}(B)$ and $\partial D \cap \mathrm{int}(B) = K \cap \mathrm{int}(B)$. Prove that K is composite if and only if there exists a nontrivial splitting sphere of K.

2 (a) For any field F of characteristic not 2, explain how the canonical isomorphism between $\mathcal{AC}_F$ and $\mathcal{W}_F$ is defined (in both directions). For any irreducible symmetric Laurent polynomial $\delta \in F[t, t^{-1}]$, explain how the projection $\mathcal{W}_F \to \mathcal{W}_F^\delta$ is defined. Also, in the case $F = \mathbb{R}$, explain how the isomorphism $\mathcal{W}_\mathbb{R}^\delta \simeq 2\mathbb{Z}$ is defined and how it is related to Levine–Tristram signature of knots.

(b) Consider the torus knot $T_{2,5}$. For each symmetric irreducible Laurent polynomial $\delta \in \mathbb{R}[t, t^{-1}]$, consider the image $[T_{2,5}]_\delta$ under the map

$$\mathcal{C} \to \mathcal{AC} \to \mathcal{AC}_\mathbb{Q} \simeq \mathcal{W}_\mathbb{Q} \to \mathcal{W}_\mathbb{R} \to \mathcal{W}_\mathbb{R}^\delta.$$

For which polynomials δ do we have $[T_{2,5}]_\delta \neq 0$? Among them, is there any δ such that $[T_{2,5}]_\delta$ generates $\mathcal{W}_\mathbb{R}^\delta$?

You can use the fact that the knot $T_{2,5}$ has a genus 2 Seifert surface whose Seifert matrix is given by

$$A = \begin{pmatrix} -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix},$$

and for any $\omega \in S^1$, we have

$$\det[(1-\omega)A + (1-\bar{\omega})A^T] = (\omega - 1)^2(\bar{\omega} - 1)^2(\omega - \alpha)(\bar{\omega} - \alpha)(\omega - \alpha^3)(\bar{\omega} - \alpha^3)$$

where $\alpha = e^{\frac{\pi i}{5}}$.

(c) Let K be a positive amphichiral knot satisfying $\mathrm{Arf}(K) = 1$. Show that for every prime p satisfying $p = 3 \pmod 4$, the exponent of p in the prime factorization of $\Delta_K(-1)$ is even.

(d) Consider the knot K as shown in Figure 1. What is its order in the algebraic concordance group? Does it have order 4 in $\mathcal{W}_{\mathbb{Q}_3}$? How about $\mathcal{W}_{\mathbb{Q}_{11}}$?

You can use the fact that K has a genus 2 Seifert surface whose Seifert matrix is given by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 \end{pmatrix}.$$

(e) Prove that, for any knot K, its $(2,5)$-cable $K_{2,5}$ does not bound a punctured torus in B^4.

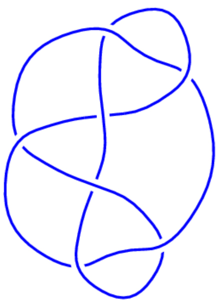

Figure 1: A knot with 7 crossings.

3 You can use the fact that the two knot diagrams given in Figure 2 represent the same knot; we call it the *Stevedore knot*.

(a) What is the definition of a *slice knot*? What is the definition of a *doubly slice knot*? What are the definition of the *slice genus* and the *double slice genus* of knots?

(b) Given any knot K and an integer m, define the *m-twist spun knot* $\mathrm{Tw}_m(K)$ of K. State Zeeman's theorem for the case $m = \pm 1$. Draw a banded-unlink diagram representing the 0-twist spun knot of the right-handed trefoil $K = T_{2,3}$.

(c) Show that the Stevedore knot is not doubly slice and its Levine–Tristram signature vanishes identically. Deduce that there exist knots that are not doubly slice but have vanishing Levine–Tristram signatures.

(d) Show that the Stevedore knot is slice. Use it to draw an explicit banded-unlink diagram of a 2-knot, consisting of one birth, two saddles, and one death, that is not unknotted.

(e) Let K be a slice knot and $\nu(K)$ be its tubular neighborhood. Choose a Seifert longitude $\nu_K \subset \partial\nu(K)$, so that $K \cup \nu_K$ is a 2-component link, which we denote by $K_{2,0}$. Show that $K_{2,0}$ can be oriented so that it is doubly slice with respect to the trivial coloring.

Hint: If K bounds a slice disk, then $K_{2,0}$ bounds a pair of disjoint disks in B^4.

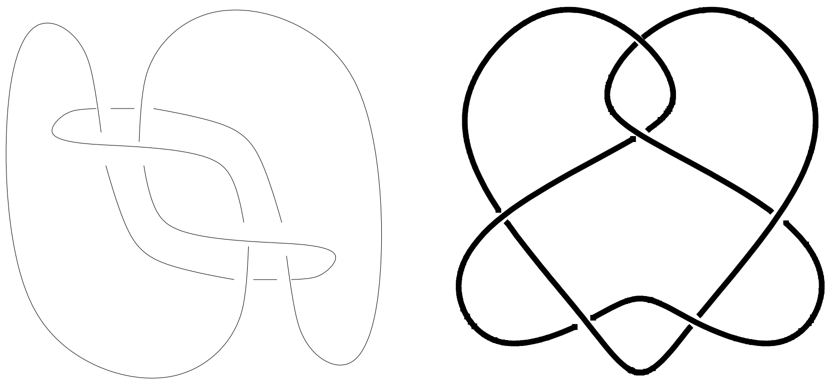

Figure 2: Two knot diagrams for the Stevedore knot.

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