UNIVERSITY OF CAMBRIDGE

MAMA/111, NST3AS/111, MAAS/111

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 2:00pm to 5:00pm

PAPER 111

COXETER GROUPS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let V be a finite dimensional real inner product space, and let Φ be a root system in V, with reflection group $W = W(\Phi)$.

(a) Define a fundamental system Δ in Φ and the associated positive system Π.

(b) Show that if $\alpha \in \Delta$ then $s_\alpha(\Pi) = (\Pi \sqcup \{-\alpha\}) \backslash \{\alpha\}$.

(c) Prove that if $\alpha \in \Delta$ and $w \in W$ then $w(\alpha) \in \Pi$ if and only if $\ell(ws_\alpha) > \ell(w)$.

(d) Show that $\ell(w) = |\{\beta \in \Pi : w(\beta) \in -\Pi\}|$ for all $w \in W$.

(e) Prove that there is a unique element $w_0 \in W$ of maximal length, and write down its length in terms of $|\Phi|$.

[Hint: In part (e), you may assume without proof that W permutes the set of fundamental systems transitively.]

2 Let (W, I, M) be a Coxeter system, $n := |I| < \infty$, and for each $i, j \in I$ let $m_{i,j}$ be the (i,j)-entry of M. We assume that $m_{i,j} < \infty$.

Define the *Gram matrix* of W to be the $n \times n$ real matrix $G(W)$ with (i,j)-entry $-2\cos\left(\frac{\pi}{m_{i,j}}\right)$.

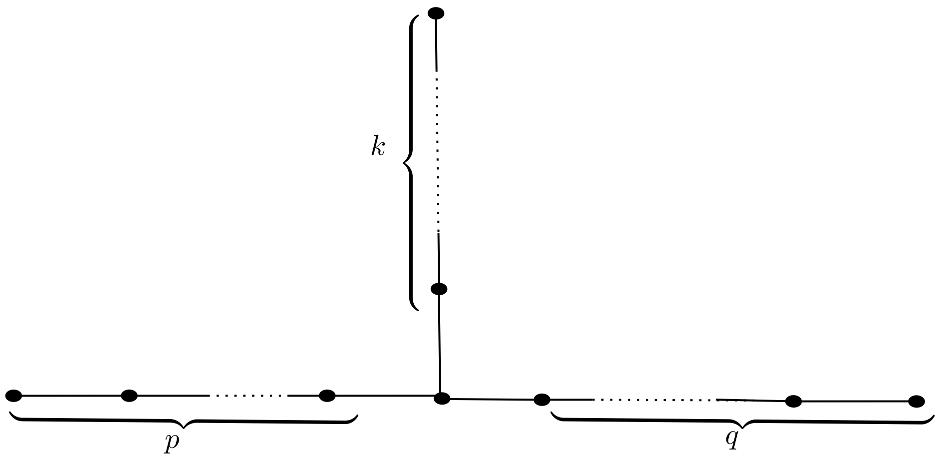
(a) Define the associated bilinear form and the geometric representation of (W, I, M) in terms of $G(W)$.

(b) Define the Coxeter graph of (W, I, M), and state what it means for W to be irreducible.

Suppose the vertex set I decomposes as $I = I_1 \sqcup I_2$, and that there exists $i \in I_1$, $j \in I_2$ such that no edge in the graph links I_1 to $I_2 \backslash \{j\}$ or I_2 to $I_1 \backslash \{i\}$. Let W_1, W_2, W_1', W_2' be the associated Coxeter groups of I_1, I_2, $I_1 \backslash \{i\}$ and $I_2 \backslash \{j\}$ respectively.

(c) Describe $G(W)$ in terms of $G(W_1), G(W_2), G(W_1'), G(W_2')$, and deduce a formula for $\det(G(W))$.

For each $p, q, k \geqslant 0$, $p \neq 0$, we say the graph below has type $E(p, q, k)$.



(d) Using the formula from part c, calculate the determinant of the Gram matrix of $E(p, q, k)$. For which values of p, q, k is the associated bilinear form non-degenerate?

[Hint: You may assume that the determinant of the Gram matrix in type A_n is $n + 1$.]

3 Throughout this question, let G be a group, and let k be a field.

(a) Define the generic Hecke algebra $\mathcal{H}_{\text{gen}}(W, \{a_i : i \in I\})$ over a Coxeter system (W, I, M). How many parameters define this algebra in type A_n?

(b) Define a BN-pair in G, its associated Weyl group W and its Iwahori-Hecke algebra $\mathcal{H}_k(G, B)$. Describe how to realise this algebra as a specialisation of the generic algebra.

Now suppose G is finite, (B, N, S) defines a BN-pair in G, and suppose that B is a q-group (i.e. a group of order q^d for some $d \in \mathbb{N}$) and k has characteristic p for some primes $p, q > 0$.

(c) If $q \equiv 1 \pmod{p}$, show that $\mathcal{H}_k(G, B) \cong k[W]$.

(d) If $q = p$, prove that $\mathcal{H} = \mathcal{H}_k(G, B)$ contains a non-zero element X such that $X^2 = 0$ and $\text{Span}_k\{X\}$ is left $\mathcal{H}$-invariant. Hence or otherwise conclude that $\mathcal{H}$ is not semisimple.

[Hint: Recall that if W is finite then there is a unique element $w_0 \in W$ of maximal length.]

4 Let (W, I, M) be a Coxeter system with generators $\{x_i : i \in I\}$, and we assume $I = \{1, \ldots, n\}$. A *Coxeter element* of W is defined to be a conjugate of $c := x_1 \cdots x_n$. The *Coxeter number* of W is the order of any Coxeter element.

(a) Calculate the Coxeter numbers of D_{2n}, S_n, SS_n and ESS_n.

(b) Let $V = \mathrm{Span}_{\mathbb{R}}\{e_i : i \in I\}$, and let $\sigma : W \to GL(V)$ be the geometric representation. Setting $\beta_i := \sigma(x_1 \cdots x_{i-1})(e_i)$ for $i \leqslant n$, prove that

$$e_j = \beta_j - \sum_{i<j} 2\cos\left(\frac{\pi}{m_{i,j}}\right)\beta_i$$

and

$$\sigma(c)(e_j) = \beta_j + \sum_{i \geqslant j} 2\cos\left(\frac{\pi}{m_{i,j}}\right)\beta_i$$

(c) Let C be the matrix representing $\sigma(c)$ with respect to the basis $\{e_i : i \in I\}$. Use part (b) to find an upper triangular matrix U and a lower triangular matrix L such that $\det(tI - C) = \det(tU - L)$.

(d) Prove that Coxeter elements fix no non-zero vectors in V whenever W is finite or hyperbolic, and show that they always fix some non-zero vector in V whenever W is affine.

END OF PAPER