UNIVERSITY OF CAMBRIDGE

MAMA/107, NST3AS/107, MAAS/107

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 11 June 2026 2:00pm to 5:00pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
Questions 1 and 2 are worth 20 marks each.
Question 3 is worth 25 marks. Question 4 is worth 35 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $u \in C^\infty(\Omega)$ be a harmonic function on a domain $\Omega \subseteq \mathbb{R}^n$.

(a) State the mean value property for u. Use this to infer that u satisfies the strong maximum principle.

(b) Prove that if $\overline{B_r(x_0)} \subset \Omega$ then

$$|Du(x_0)| \leqslant \frac{C(n)}{r} \sup_{B_r(x_0)} |u|.$$

(c) Prove the following version of Liouville's Theorem: if $u \in C^\infty(\mathbb{R}^n)$ is harmonic, and if there are $\alpha \in (0,1)$, $k \in \mathbb{N}$ and $C > 0$ such that $|u(x)| \leqslant C(1 + |x|^{k+\alpha})$ for all $x \in \mathbb{R}^n$, then u is a polynomial of degree at most k.

2 Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $\alpha \in (0,1)$ and $\lambda > 0$. Suppose that $a_{ij} \in C^\alpha(\overline{\Omega})$ are coefficients satisfying $a_{ij}(x) = a_{ji}(x)$ and $a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. Consider a function $f \in C^\alpha(\overline{\Omega})$.

(a) Show that there exists a solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} a_{ij}\partial_{ij}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

[You may assume that this holds for $a_{ij} \equiv \delta_{ij}$, and any estimate proved in the lectures, if properly stated.]

(b) Let now $\Omega' \subset \Omega$ be another smooth bounded domain, with $\Omega' \neq \Omega$, and suppose that there exists a point $y_0 \in \partial\Omega \cap \partial\Omega'$, that is, the boundaries of the domains touch at y_0. Let $v \in C^2(\overline{\Omega'})$ solve

$$\begin{cases} a_{ij}\partial_{ij}v = f & \text{in } \Omega' \\ v = 0 & \text{on } \partial\Omega', \end{cases}$$

and let $\nu \in \mathbb{S}^{n-1}$ denote the common outer unit normal vector to Ω and Ω' at y_0. Suppose in addition that $f > 0$ in Ω. Show that $\mathrm{D}_\nu u(y_0) \neq \mathrm{D}_\nu v(y_0)$.

[You may assume any results from the lectures if properly stated.]

3 Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain.

(a) Define the Campanato spaces $\mathcal{L}^{p,\mu}(\Omega)$ for $1 \leqslant p < \infty$ and $\mu \geqslant 0$ and state their relation to the Hölder spaces $C^\alpha(\Omega)$.

(b) Show the following variant of the Schauder regularity theorem for divergence equations with Hölder continuous coefficients: if $\alpha \in (0,1)$, $a_{ij} \in C^\alpha(B_1)$ satisfy $a_{ij}(x) = a_{ji}(x)$ and $a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2$ for all $x \in B_1$ and $\xi \in \mathbb{R}^n$ (where $\lambda > 0$ is a constant), and $u \in C^1(\overline{B_1})$ solves $\partial_i(a_{ij}\partial_j u) = 0$ in the weak sense, that is,

$$\int_{B_1} a_{ij}(x)\partial_i\varphi(x)\partial_j u(x)\,\mathrm{d}x = 0 \qquad \forall \varphi \in C_c^\infty(B_1),$$

then $u \in C^{1,\alpha}(B_{1/2})$.

[Hint: use the fact that, since $u \in C^1(\overline{B_1})$, $|\mathrm{D}u| \leqslant L$ for some constant $L > 0$. You may use the decay estimates for solutions to constant coefficient equations with zero right hand side and the iteration lemma seen in the lectures, provided that you state them correctly. You may also use without proof the estimate

$$\int_\Omega |\nabla w|^2 \leqslant \frac{1}{\lambda^2}\int_\Omega |F|^2$$

satisfied by $w \in W_0^{1,2}(\Omega)$ solving $\operatorname{div}(B\nabla w) = \operatorname{div}(F)$, where B is constant and $\langle B\xi, \xi\rangle \geqslant \lambda|\xi|^2$.]

4 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $F \in C^2(\mathbb{R}^n)$ strictly convex, and $\mathcal{F}[u] := \int_\Omega F(Du(x))\,dx$ the corresponding functional. Moreover let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a given Lipschitz function.

(a) Denote $\mathrm{Lip}_g(\Omega, k) = \{u \in \mathrm{Lip}(\Omega) : u = g \text{ on } \partial\Omega, \mathrm{Lip}(u) \leqslant k\}$. Show that $\mathcal{F}$ attains its infimum in this class at an element u.

(b) Let u be the function obtained in (a). Show that, if $\mathrm{Lip}(u) < k$, then u is a minimiser of $\mathcal{F}$ in $\mathrm{Lip}_g(\Omega) = \{u \in \mathrm{Lip}(\Omega) : u = g \text{ on } \partial\Omega\}$.

We say that the function g satisfies the bounded slope condition with constant $K > 0$ if, for every $x_0 \in \partial\Omega$, there are two affine functions $b^+_{x_0}, b^-_{x_0}$ with Lipschitz constant at most K and such that

$$b^-_{x_0} \leqslant g \leqslant b^+_{x_0} \text{ on } \partial\Omega,$$

with equality at x_0. For short we say that g has $(BSC)_K$ if this holds.

(c) Show that, if $g \in C^2(\mathbb{R}^n)$ and $\Omega = B_1(0)$ then g has $(BSC)_K$ for some K.

[Hint: use the identity $|x - x_0|^2 = 2x_0 \cdot (x_0 - x)$ for $x, x_0 \in \partial\Omega$ together with Taylor's theorem.]

Now let $k > K$ and let u be a minimiser of $\mathcal{F}$ restricted to $\mathrm{Lip}_g(\Omega, k)$ as above.

(d) Show that if g has $(BSC)_K$ then $\mathrm{Lip}(u) \leqslant K$.

[You may assume that the comparison principle holds for minimisers of $\mathcal{F}$ restricted to the class $\mathrm{Lip}_g(\Omega, k)$. You may also use without proof the fact that, for a minimiser u of $\mathcal{F}$ restricted to $\mathrm{Lip}_g(\Omega, k)$, if

$$\forall x \in \Omega\, \forall y \in \partial\Omega \quad |u(x) - u(y)| \leqslant K|x - y|,$$

then $\mathrm{Lip}(u) \leqslant K$ holds.]

(e) Let $F(\xi) = \sqrt{1 + |\xi|^2}$ be the area integrand, $\Omega = B_1(0)$ and $g \in C^2(\mathbb{R}^n)$. Show that $\mathcal{F}$ has a minimiser in $\mathrm{Lip}_g(\Omega)$ and it is smooth in the interior of Ω.

[You may assume results from lectures if clearly stated.]

END OF PAPER