UNIVERSITY OF CAMBRIDGE

MAMA/106, NST3AS/106, MAAS/106

MAT3

MATHEMATICAL TRIPOS Part III

Friday 5 June 2026 9:00am to 12:00pm

PAPER 106

FUNCTIONAL ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the notions of *(Schauder) basis* and *basis constant*. [You do not need to prove that the basis constant is finite.]

Let $(e_n)_{n=1}^{\infty}$ be a basis of a real Banach space X. Show that X is isomorphic to the space Y of real sequences defined as follows:

$$Y = \left\{ (a_n) : \left(\sum_{k=1}^{n} a_k e_k \right)_n \text{ is convergent in } X \right\}$$

with norm $\|(a_n)\| = \sup_n \left\| \sum_{k=1}^{n} a_k e_k \right\|$.

Define the *dual basis* (e_n^*) of (e_n) and show that it is a basic sequence in X^*. Show also that if P_k denotes the k^{th} basis projection of (e_n), then $P_k^*(x^*) \xrightarrow{w^*} x^*$ for all $x^* \in X^*$. [Standard criteria for bases can be used without proof.]

Show that if X is reflexive, then (e_n^*) is a basis of X^*. Does the converse hold? Justify your answer.

Show that if (e_n^*) is a basis of X^*, then X^{**} is isomorphic to the space Z of real sequences defined as follows:

$$Z = \left\{ (a_n) : \sup_n \left\| \sum_{k=1}^{n} a_k e_k \right\| < \infty \right\}$$

with norm $\|(a_n)\| = \sup_n \left\| \sum_{k=1}^{n} a_k e_k \right\|$. [Results about the weak-star topology may be used without proof.]

2 (a) Define the *spectrum* $\sigma_A(x)$ of an element x of a (complex) algebra A. Prove that if A is a Banach algebra, then $\sigma_A(x)$ is a non-empty compact subset of $\mathbb{C}$. [Elementary properties of the group of invertible elements of a unital Banach algebra can be assumed without proof.]

(b) What is a *character* on an algebra A? What is the *character space* Φ_A of A?

(c) Let A be a Banach algebra. Show that characters on A are continuous. Define the *Gelfand topology* on Φ_A. Show that if A is a unital Banach algebra, then Φ_A is compact in the Gelfand topology.

(d) Describe, without proof, the character space of the Banach algebra $C(K)$ of complex-valued continuous functions on the compact Hausdorff space K.

(e) Let $A = C_0(\mathbb{R})$, the algebra of complex-valued continuous functions f on $\mathbb{R}$ that vanish at infinity: $f(x) \to 0$ as $|x| \to \infty$. Prove that $\Phi_A = \{\delta_x : x \in \mathbb{R}\}$, where $\delta_x(f) = f(x)$ ($x \in \mathbb{R}$, $f \in A$). [You may assume the existence of a homeomorphism $\tau \colon \mathbb{R} \to \mathbb{T} \setminus \{1\}$ with $\tau(x) \to 1$ as $|x| \to \infty$.]

(f) Let $A = C(\mathbb{R})$, the algebra of complex-valued continuous functions on $\mathbb{R}$.

(i) Prove that $\Phi_A = \{\delta_x : x \in \mathbb{R}\}$. [Hint: Given $\varphi \in \Phi_A$, first show that on $C_0(\mathbb{R})$, $\varphi = \delta_x$ for some $x \in \mathbb{R}$. It might be helpful to consider the functions t, $\frac{1}{t^2+1}$ and $\frac{t}{t^2+1}$. Then for $h \in A$ with $h \geqslant 0$ on $\mathbb{R}$, consider $\frac{1}{h(t)+t^2+1}$.]

(ii) Prove that there is no complete algebra norm on A.

(iii) Assume now that A has an (incomplete) algebra norm $\|\cdot\|$, and let B be the completion of A. Show that

$$K = \{x \in \mathbb{R} : \delta_x \text{ is continuous with respect to } \|\cdot\|\}$$

is homeomorphic to the character space Φ_B of B.

(iv) Deduce that $C(\mathbb{R})$ cannot be given any algebra norm. [Hint: Consider suitable non-zero functions $a, b \in A$ with $ab = 0$.]

3 Let K be a subset of a Banach space X. Prove that if K is bounded, $0 \notin \overline{K}$ and $0 \in \overline{K}^w$, then K contains a basic sequence. [Standard criteria for bases can be used without proof.] State, but do not prove, a generalisation of this result in which the weak topology is replaced by a weaker topology. Give an example of such a topology in the case X is not reflexive, and explain *briefly* why the example can be chosen to be strictly weaker than the weak topology.

Prove the Eberlein–Šmulian theorem: K is relatively weakly compact in X if and only if every sequence in K has a subsequence that converges weakly in X.

Show that if K is weakly sequentially compact and $x \in \overline{K}^w$, then there is a sequence in K converging weakly to x. Deduce that K is weakly compact if and only if K is weakly sequentially compact.

[*Standard results about the weak topology and the weak-star topology may be used without proof.*]

4 Let $T\colon X \to Y$ be a linear map between Banach spaces. What does it mean to say that T is *weakly compact*? Show that if T is weakly compact, then T is bounded.

Let $T \in \mathcal{B}(X,Y)$. Show that T is weakly compact if and only if T^* is weakly compact. Show also that the set $\mathcal{W}(X,Y)$ of all weakly compact operators from X to Y is a closed subspace of $\mathcal{B}(X,Y)$ with the ideal property.

State and prove the Krein–Šmulian theorem.

Show that if X or Y is reflexive, then $\mathcal{B}(X,Y) = \mathcal{W}(X,Y)$. Show that if $X = \ell_1$ and Y is not reflexive, then $\mathcal{B}(X,Y) \neq \mathcal{W}(X,Y)$. Must it always be the case that $\mathcal{B}(X,Y) \neq \mathcal{W}(X,Y)$ whenever X and Y are both non-reflexive?

END OF PAPER