UNIVERSITY OF CAMBRIDGE

MAMA/105, NST3AS/105, MAAS/105

MAT3

MATHEMATICAL TRIPOS Part III

Monday 15 June 2026 9:00am to 12:00pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the following PDE for an unknown function $u = u(t, r, z)$, posed on $\mathbb{R} \times (0, \infty) \times \mathbb{R}$

$$u_{tt} - \big(1 + u_z\big)^2 \Big(u_{rr} + \frac{1}{r} u_r + u_{zz}\Big) = 0. \tag{X}$$

Let $\phi : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be real analytic and set

$$\Sigma = \{(t, r, z) \in \mathbb{R} \times (0, \infty) \times \mathbb{R} : \phi(t, r, z) = 0\}.$$

Assume $\nabla_{(t,r,z)}\phi \neq 0$ on Σ. Denote the normal vector $n := \nabla_{(t,r,z)}\phi = (\phi_t, \phi_r, \phi_z)$ and the associated unit normal $N := \frac{n}{|n|}$.

(a) Assuming prescribed values for u_z on Σ, write the principal symbol of (X) and determine the non-characteristic condition for Σ in terms of ϕ and the boundary values $u_z|_\Sigma$.

(b) State, for arbitrary $j \in \mathbb{N}$, the definition of the *j-th normal derivative* $\partial_N^j u(x)$ of u at $x \in \Sigma$. Assuming u is C^1 in a neighborhood of Σ, compute $\partial_N u$ on Σ explicitly in terms of u and ϕ.

(c) Now let $\phi(t, r, z) = t - \psi(r, z)$ with $\psi : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ real analytic, and denote the associated hypersurface by $\Sigma = \Sigma_\psi$. Assume that Cauchy data on Σ_ψ are prescribed by

$$u(\psi(r, z), r, z) = 0, \qquad (\partial_N u)(\psi(r, z), r, z) = g(r, z), \tag{D}$$

where $g = g(r, z)$ is real analytic on $(0, \infty) \times \mathbb{R}$. Write the non-characteristic condition in terms of ψ and g.

(d) State the *Cauchy–Kovalevskaya theorem* for k-th order quasilinear scalar PDEs with Cauchy data on a hypersurface Σ including clearly all hypotheses and conclusions.

(e) Define $\psi_0 : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by $\psi_0(r, z) = \frac{1}{2}(r^2 + z^2)$, and denote the associated hypersurface by Σ_{ψ_0}. Prescribe

$$u(\psi_0(r, z), r, z) = 0, \qquad (\partial_N u)(\psi_0(r, z), r, z) = \sqrt{1 + r^2 + z^2}. \tag{D$_0$}$$

Determine for which points $(t_0, r_0, z_0) \in \Sigma_{\psi_0}$ we can guarantee existence and uniqueness of a local real analytic solution to (X) satisfying (D$_0$).

2 Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded, connected domain with C^1 boundary. Assume that Γ_1, Γ_2 are disjoint, relatively open subsets of $\partial\mathcal{U}$, $\partial\mathcal{U} = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ and suppose that Γ_1 has positive $(n-1)$-dimensional measure: $|\Gamma_1| > 0$. Consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{U}, \\ u = 0 & \text{on } \Gamma_1, \\ \partial_\nu u = 0 & \text{on } \Gamma_2, \end{cases} \tag{P}$$

where $f \in L^2(\mathcal{U})$ and $\partial_\nu u$ denotes the outward normal derivative on $\partial\mathcal{U}$. Define

$$V := \{v \in H^1(\mathcal{U}) : v|_{\Gamma_1} = 0 \text{ in the sense of the trace}\}.$$

(a) Prove that there exists a constant $C > 0$, depending only on $\mathcal{U}$ and Γ_1, such that

$$\|v\|_{L^2(\mathcal{U})} \leqslant C\|\nabla v\|_{L^2(\mathcal{U})} \quad \text{for all } v \in V.$$

Show that $\|\nabla v\|_{L^2(\mathcal{U})}$ defines a norm on V equivalent to the H^1 norm restricted to V.

[Hint: Suppose for a contradiction that there exist a sequence $\{v_k\}_k \subset V$ such that $\|v_k\|_{L^2(\mathcal{U})} > k\|\nabla v_k\|_{L^2(\mathcal{U})}$. Also, recall that any linear closed subspace of $H^1(\mathcal{U})$ is weakly closed.]

(b) Using the space V above, derive the weak formulation of problem (P). That is, write the problem in the form: find $u \in V$ such that

$$B(u, v) = \ell(v) \quad \text{for all } v \in V,$$

for some bilinear form $B(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ that you must identify explicitly.

(c) Assume now that $\partial\mathcal{U}$ is of class C^2 and $f \in C(\overline{\mathcal{U}})$. Let $u \in C^2(\overline{\mathcal{U}}) \cap V$ be a weak solution in the sense of part (b). Show that u is a classical solution, namely u solves all the equations in (P) pointwise.

(d) Show that $(V, \|\cdot\|_V)$ with $\|v\|_V := \|\nabla v\|_{L^2(\mathcal{U})}$ is a Hilbert space.

(e) Show that there exists a unique $u \in V$ solving the weak formulation of (P). Moreover, show that there exists $C > 0$ depending only on $\mathcal{U}$ and Γ_1 such that $\|u\|_V \leqslant C\|f\|_{L^2(\mathcal{U})}$.

3

(a) Consider the following transport equation together with the initial datum u_0:

$$\partial_t u(t,x) + x\,\partial_x u(t,x) = 0, \qquad t > 0,\ x \in \mathbb{R}, \qquad u(0,x) = u_0(x). \tag{L}$$

Define the *characteristic curves* $Z_{s,t}(x)$ for (L) passing from x at initial time s and compute them explicitly. If $u_0 \in C^1(\mathbb{R})$, use the characteristic curves to provide a classical solution to (L).

(b) Let $u_0 \in L^\infty(\mathbb{R})$. Using test functions $\varphi \in C^1_c([0,\infty)\times\mathbb{R})$, define what it means for $u \in L^\infty((0,\infty)\times\mathbb{R})$ to be a *weak solution of* (L) with initial datum u_0. Then show that if $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u \in C^1([0,\infty)\times\mathbb{R}) \cap L^\infty((0,\infty)\times\mathbb{R})$ is a weak solution, then u is a classical solution.

(c) Let $u \in L^\infty((0,\infty)\times\mathbb{R})$ be a weak solution of (L) with $u_0 \equiv 0$. Let $\psi \in C^1_c([0,\infty)\times\mathbb{R})$. Show that there exists $\phi \in C^1_c([0,\infty)\times\mathbb{R})$ such that

$$\partial_t \phi + x\,\partial_x \phi + \phi = \psi \qquad \text{on } (0,\infty)\times\mathbb{R}. \tag{A}$$

Deduce uniqueness of weak solutions to (L) in $L^\infty((0,\infty)\times\mathbb{R})$.

[Hint: Fix t, x and set $h(r) = \phi(r, Z_{t,r}(x))$ to reduce (A) *to an ODE.]*

(d) Let $f \in C^2(\mathbb{R})$ with $f'' \not\equiv 0$, let $a \in C^1([0,\infty))$ with $a(\cdot) > \varepsilon$ for some $\varepsilon > 0$, and let $u_0 \in C^1_c(\mathbb{R})$. Consider the following transport equation together with the initial datum u_0:

$$\partial_t u + a(t)\,\partial_x\big(f(u)\big) = 0, \qquad u(0,x) = u_0(x). \tag{N$_a$}$$

Assume there exists $x_0 \in \mathbb{R}$ with $f''(u_0(x_0))\,u_0'(x_0) < 0$. Using characteristics, show that for any classical C^1 solution u of (N$_a$) there exists a first time $T_* > 0$ such that $\|\partial_x u(t,\cdot)\|_{L^\infty(\mathbb{R})} \to +\infty$ as $t \to T_*$, and compute T_* in terms of a, u_0 and f.

END OF PAPER