UNIVERSITY OF CAMBRIDGE

MAMA/102, NST3AS/102, MAAS/102

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 10 June 2026 9:00am to 12:00pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathfrak{g}$ be the Lie algebra over $\mathbb{C}$ with basis a, b, c, and brackets $[a, b] = c, [c, a] = [c, b] = 0$.

i) Show that $\mathfrak{g}$ is nilpotent.

ii) Let V be an irreducible finite dimensional representation of $\mathfrak{g}$. Prove directly that c acts as 0.

Classify all irreducible finite dimensional representations of $\mathfrak{g}$.

iii) Define the trace form $(,)_V$ for a finite dimensional representation V of a Lie algebra.

Prove that the trace form of any finite dimensional representation of the Lie algebra $\mathfrak{g}$ above is degenerate. [*You may not quote any theorems from lectures*].

iv) Give an example of a faithful representation of $\mathfrak{g}$ which is irreducible. [*You must prove that it is irreducible.*]

2 Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\mathfrak{t}$ a maximal torus, R^+ a choice of positive roots.

(i) Define the Verma module M_λ, $\lambda \in \mathfrak{t}^*$, and compute its character $ch\, M_\lambda$, carefully stating any theorems needed.

(ii) Show that if V is a submodule of M_λ, then V has a singular vector of weight μ; show $|\mu + \rho| = |\lambda + \rho|$.

(iii) Now let $\mathfrak{g} = \mathfrak{sl}_2$. For every $\lambda \in \mathbb{C}$, determine whether M_λ is irreducible. If M_λ is not irreducible, determine all the non-trivial submodules of M_λ.

3 Let
$$\mathfrak{g} = \mathfrak{sp}_{2n} = \left\{A \in Mat_{2n} | AJ + JA^T = 0\right\},$$

where $J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$, and let $\mathfrak{t}$ = diagonal matrices in $\mathfrak{g}$.

(i) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those occurring in upper triangular matrices. Write down the positive roots R^+, the simple roots Π, the highest root θ, and the fundamental weights. Write down ρ.

Write the root lattice P, the weight lattice Q, and determine Q/P.

Draw the Dynkin diagram and label it by simple roots. Draw the extended Dynkin diagram.

[You do not need to prove your answers.]

(ii) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

(iii) Show that $\mathfrak{sp}_2 = \mathfrak{sl}_2$.

4 (i) State the Weyl character formula, the Weyl dimension formula, and the q-character formula, briefly defining the notation you use.

(ii) Draw the root system of G_2, and the fundamental weights ω_1, ω_2.

Let ω_1 denote the shorter fundamental weight.

Compute the q-character of the representation with highest weight ω_1, and decompose it as a representation of the principal $\mathfrak{sl}_2$.

Now do the same for the representation with highest weight ω_2.

5 Let $\mathfrak{g} = \mathfrak{so}_5$, ω_1 and ω_2 the fundamental weights, the notation chosen so that $V = \mathbb{C}^5$ is the representation with highest weight ω_2.

Draw the crystal of V, briefly justifying your answer.

Draw the crystal of $V \otimes V$. Hence determine the crystals of $\wedge^2 V$ and $S^2 V$ and their decomposition into irreducibles.

Let L be the representation with highest weight ω_1. Draw the crystal of L, briefly justifying your answer.

Show that L does not occur as a summand of $V^{\otimes n}$ for any n.

END OF PAPER