UNIVERSITY OF CAMBRIDGE

MAMA/101, NST3AS/101, MAAS/101

MAT3

MATHEMATICAL TRIPOS Part III

Monday 8 June 2026 9:00am to 12:00pm

PAPER 101

COMMUTATIVE ALGEBRA

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the tensor-Hom adjunction isomorphism and describe the isomorphism explicitly.

(b) Prove that this isomorphism is natural in the left argument.

(c) A non-zero R-module M is called a *brick* if every R-module endomorphism $f\colon M \to M$ is either zero or an isomorphism.

(i) Is every brick a simple module? Justify your answer.

(ii) Show that if L, N are two bricks with $\mathrm{Hom}_R(L, N) \cong \mathrm{Hom}_R(N, L) \cong 0$, then if $0 \to L \to M \to N \to 0$ is a non-split short exact sequence, then M is also a brick.

[*You may use without proof that if f and h are isomorphisms in the following commutative diagram, then so is g.*]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

2

(a) Given an ideal I of a ring R, what is a minimal primary decomposition of I?

(b) State and prove the Second Uniqueness Theorem of Primary Decomposition.

(c) Let R be a Noetherian ring with M a finitely generated R-module.

A submodule N of M is called *primary* if $N \neq M$ and for any $r \in R$ and $m \in M \setminus N$, if $rm \in N$, then there exists a positive integer k such that $r^k M \subseteq N$.

If $\mathrm{Ann}(M/N)$ is a primary ideal, is N a primary submodule of M? Justify your answer.

(d) Let I be a primary ideal of R, with t a formal variable over R.

Show that I^e is a primary ideal of $R[t]$, where extension is along the natural inclusion $R \to R[t]$.

3

(a) State the Going-Down Theorem.

(b) Suppose that $R \subseteq A$ are integral domains satisfying the assumptions of the Going-Down Theorem.

Let $\mathfrak{p}$ be a prime ideal of R, with $\mathfrak{p}^e$ its extension along the inclusion $R \to A$.

Show that if $y \in \mathfrak{p}^e$, then the minimal polynomial of y over $\operatorname{Frac} R$ has non-leading coefficients in $\mathfrak{p}$.

(c) Let R be a ring, with A and A' two R-algebras. Show that if A is integral over R, then $A \otimes_R A'$ is integral over A'.

(d) Let R be a ring, with A an R-algebra and $p \in A[t_1, \ldots, t_n]$ a polynomial. Show that p is integral over $R[t_1, \ldots, t_n]$ if and only if its coefficients are integral over R.

4

(a) How is the inverse limit of an inverse system $((N_i)_{i \in I}, (g_{ij})_{i \leqslant j})$ defined?

(b) State the definition of equivalence for a pair of filtrations of an R-module M. Show that any two stable I-filtrations of an R-module M are equivalent.

(c) A *filtered ring* is a ring R with a filtration $(R_n)_{n \geqslant 0}$ of R as an R-module such that $R_m R_n \subseteq R_{m+n}$ for all m, n.

Given a module M over a filtered ring R as above, its *induced* filtration is $(M_n)_{n \geqslant 0}$ where $M_n := R_n M$.

Now let N be a submodule of M.

(i) Is the filtration $(M_n \cap N)_{n \geqslant 0}$ always the induced filtration of N? Justify your answer.

(ii) Is the filtration $((M_n + N)/N)_{n \geqslant 0}$ always the induced filtration of M/N? Justify your answer.

(d) Now let R be a Noetherian ring. Let $I, I' \trianglelefteq R$ with $I \subseteq I'$. An ideal $J \trianglelefteq R$ is called a *separating ideal* for (I, I') if $I = I' \cap (I + J)$.

Now let J be a separating ideal for (I, I') as above. Show that if $r \in \sqrt{(I : I')}$, then there exists a positive integer m satisfying $I = I' \cap (I + J + (r^m))$.

5

(a) Let $(R, \mathfrak{m})$ be a Noetherian local ring.

(i) Define the characteristic polynomial of an $\mathfrak{m}$-primary ideal I.

(ii) Show that this polynomial has degree bounded by the number of generators of I. [You may use standard results on Hilbert polynomials.]

(b) Let K be a field. Consider the graded ring

$$R = \frac{K[x, y, t_1, \ldots, t_n]}{(x^k - y^k)},$$

where $\deg x = \deg y = 1$, $\deg t_i = i$ and K is in degree zero.

Compute the Poincaré series of R as an R-module with respect to the additive function given by length as an R_0-module. You should justify the steps of your calculation and give your answer as a rational function expressed as a single fraction.

(c) Now let $p_1, p_2, \ldots, p_n$ be distinct prime numbers. Consider the graded ring

$$R = \frac{\frac{\mathbb{Z}}{p_1 \cdots p_n \mathbb{Z}}[t_1, \ldots t_n]}{(p_1 t_1, \ldots, p_n t_n)},$$

where t_i all have degree one and elements of $\frac{\mathbb{Z}}{p_1 \cdots p_n \mathbb{Z}}$ have degree zero.

(i) Compute the Poincaré series of R as an R-module with respect to the additive function given by length as an R_0-module. You should justify the steps of your calculation and give your answer as a rational function expressed as a single fraction.

(ii) Comment on the degree of the polynomial in the denominator of this rational function.

END OF PAPER