UNIVERSITY OF CAMBRIDGE

MAMA/359, NST3AS/359, MAAS/359

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday 17 June 2025 1:30 pm to 4:30 pm

PAPER 359

MATHEMATICAL ANALYSIS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
Question 1 carries 60 marks and question 2 carries 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\Omega = \mathbb{T}^3 := (\mathbb{R}/L\mathbb{Z})^3$ be a fundamental periodic domain in $\mathbb{R}^3$, and let

$$H = \left\{ v \in (\dot{L}^2_{\text{per}}(\Omega))^3 \middle| \quad \nabla \cdot v = 0 \right\},$$

and

$$V = \left\{ v \in (\dot{H}^1_{\text{per}}(\Omega))^3 \middle| \quad \nabla \cdot v = 0 \right\},$$

with $|\cdot|$ and $\|\cdot\|$ denoting the norms of the spaces H and V respectively. We denote by $B(u,v)$ the bilinear operator $B : V \times V \to V'$ defined by

$$\langle B(u,v), w \rangle_{V' \times V} = \int_\Omega (u(x) \cdot \nabla v(x)) \cdot w(x) dx, \quad \text{for every} \quad u, v, w \in V.$$

Recall that $D(A) = (\dot{H}^2_{\text{per}}(\Omega))^3 \cap V$ is the domain of the Stokes operator A, where $A = -\Delta$ in this setting.

Suppose that $\{\varphi_1, \varphi_2, \cdots\}$ is an orthonormal basis of the Hilbert space H consisting of eigenfunctions of the Stokes operator A, that is $A\varphi_k = \lambda_k \varphi_k$ for $k = 1, 2, \cdots$, with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, repeated according to their multiplicities. Furthermore, denote by

$$H_n = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_n\},$$

and by P_n the orthogonal projection $P_n : H \to H_n$, for $n = 1, 2, \cdots$.

Let N be a fixed positive integer. Consider the following system in $\mathbb{R}^3$, subject to periodic boundary condition with fundamental domain Ω,

$$\begin{cases} \frac{\partial}{\partial t} u - \nu \Delta u + (P_N u) \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x). \end{cases}$$

Or equivalently we consider the evolution equation

$$\frac{du}{dt} + \nu A u + B(P_N u, u) = 0, \tag{$*$}$$

with initial value $u(0) = u_0$.

(a) (i) Show that

$$\|P_N w\| \leqslant \lambda_N^{1/2} |P_N w| \quad \text{for every } w \in V,$$

(ii) and that there exists a positive scale invariant constant c, such that:

$$\|P_N w\|_{L^3(\Omega)} \leqslant c \lambda_N^{1/4} |w| \quad \text{for every } w \in V,$$

(iii) moreover, that

$$|\langle B(P_N u, v), w \rangle_{V' \times V}| \leqslant c \lambda_N^{1/4} |u| \, \|v\| \, \|w\|, \quad \text{for every } u, v, w \in V.$$

(b) Let $u_0 \in H$ and let $m > N$ be an arbitrary integer. Consider the following Galerkin approximation scheme in H_m for equation $(*)$:

$$\frac{d}{dt} u_m + \nu A u_m + P_m B(P_N u_m, u_m) = 0, \qquad (**)$$

with initial value $u_m(0) = P_m u_0$.

(i) Explain briefly why for every $T > 0$ the initial-value ODE system $(**)$ has a unique solution in the time interval $[0, T]$.

(ii) Demonstrate that for every $T > 0$, there are positive constants $K_0(T), \overline{K_1}(T)$ and $\overline{K_0'}(T)$, independent of m, such that the solution u_m of the Galerkin system $(**)$ satisfies:

$$\|u_m\|_{L^\infty((0,T);H)} \leqslant K_0(T)\,,$$

$$\|u_m\|_{L^2((0,T);V)} \leqslant \overline{K_1}(T)\,,$$

and

$$\left\| \frac{du_m}{dt} \right\|_{L^2((0,T);V')} \leqslant \overline{K_0'}(T)\,.$$

(c) Let us denote by H_{weak} the Hilbert space H endowed with its weak topology. Use the above to prove that for every $T > 0$ and every choice of initial data $u_0 \in H$ there exists a global weak solution, u, to system $(*)$ satisfying:

(i)

$$u \in C([0, T]; H_{\text{weak}}) \bigcap L^\infty((0, T); H) \bigcap L^2((0, T); V), \quad \text{and} \quad \frac{du}{dt} \in L^2((0, T); V');$$

and that $(*)$ holds in $L^2((0, T); V')$.

(ii) Moreover, show that the weak solution, u, of system $(*)$, which was established above, satisfies the energy equality:

$$\frac{1}{2} |u(t)|^2 + \nu \int_0^t \|u(\tau)\|^2 \, d\tau = \frac{1}{2} |u_0|^2\,,$$

for every $t \in [0, T]$;

(iii) and that

$$u \in C([0, T]; H).$$

(iv) Demonstrate that the weak solution is unique.

2 Let $\Omega = \mathbb{T}^3 := (\mathbb{R}/L\mathbb{Z})^3$ be a fundamental periodic domain in $\mathbb{R}^3$, and let

$$\tilde{H} = (\dot{L}^2_{\text{per}}(\Omega))^3, \quad \text{and} \quad \tilde{V} = (\dot{H}^1_{\text{per}}(\Omega))^3,$$

with $|\cdot|$ and $\|\cdot\|$ denote the norms of the spaces $\tilde{H}$ and $\tilde{V}$, respectively. Moreover, we denote by $\tilde{V}'$ the dual space of $\tilde{V}$.

We set the bilinear operator $\tilde{B}(u,v)$ to be the map $\tilde{B} : \tilde{V} \times \tilde{V} \to \tilde{V}'$ defined by the following formula:

$$\langle \tilde{B}(u,v), w \rangle_{\tilde{V}' \times \tilde{V}} = \int_\Omega [(u(x) \cdot \nabla v(x)) \cdot w(x) + \frac{1}{2}(\nabla \cdot u(x)) v(x) \cdot w(x)]\, dx, \qquad (\ddagger)$$

for every $u, v, w \in \tilde{V}$.

Moreover, let the operator $\tilde{A} = -\Delta$ with domain $D(\tilde{A}) = (\dot{H}^2_{\text{per}}(\Omega))^3$, and assume that $\tilde{A}$ can be extended uniquely to a linear operator $\tilde{A} : \tilde{V} \to \tilde{V}'$. We set $\{\psi_1, \psi_2, \cdots\}$ to be an orthonormal basis of $\tilde{H}$ consisting of eigenfunctions of the operator $\tilde{A}$, that is $\tilde{A}\psi_k = \mu_k \psi_k$ for $k = 1, 2, \cdots$, with $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$, repeated according to their multiplicities. Furthermore, set

$$\tilde{H}_n = \text{span}\{\psi_1, \psi_2, \cdots, \psi_n\},$$

and denote by $\tilde{P}_n$ the orthogonal projection $\tilde{P}_n : \tilde{H} \to \tilde{H}_n$, for $n = 1, 2, \cdots$.

Suppose $f \in \tilde{H}$ and consider the following nonlinear steady state equation in $\mathbb{R}^3$, subject to periodic boundary condition with fundamental domain Ω,

$$-\nu \Delta u + (u \cdot \nabla) u + \frac{1}{2}(\nabla \cdot u) u = f. \qquad (\dagger)$$

(a) (i) Show that the bilinear operator $\tilde{B}(u,v)$ given in equation ($\ddagger$) is well defined, and that there exists a dimensionless constant $c > 0$ such that

$$\left| \langle \tilde{B}(u,v), w \rangle_{\tilde{V}' \times \tilde{V}} \right| \leqslant c(|u|^{1/2} \|u\|^{1/2} \|v\| + |v|^{1/2} \|v\|^{1/2} \|u\|) \|w\| \leqslant c\mu_1^{-1/4} \|u\| \|v\| \|w\|,$$

$u, v, w \in \tilde{V}$.

(ii) Show that

$$\langle \tilde{B}(u,v), w \rangle_{\tilde{V}' \times \tilde{V}} = -\langle \tilde{B}(u,w), v \rangle_{\tilde{V}' \times \tilde{V}}, \quad \text{for every} \quad u, v, w \in \tilde{V}.$$

(b) Consider the following Galerkin approximation scheme in $\tilde{H}_m$ for equation ($\dagger$), for $m = 1, 2, \cdots$:

$$\nu \tilde{A} u_m + \tilde{P}_m \tilde{B}(u_m, u_m) = \tilde{P}_m f. \qquad (\dagger\dagger)$$

(i) Show that there exists $R > 0$, which depends on $|f|, \nu$ and μ_1, but is independent of m, such that all the solutions of ($\dagger\dagger$) satisfy

$$\|u_m\| \leqslant R.$$

(ii) Prove that for every $m = 1, 2, \cdots$ the nonlinear system (††) has a solution in $\tilde{H}_m$.

[Hint: You may want to consider the map $\Phi_m : \tilde{H}_m \to \tilde{H}_m$ defined by

$$\Phi_m(v) = -v - (\nu\tilde{A})^{-1}\tilde{P}_m\tilde{B}(v, v) + (\nu\tilde{A})^{-1}\tilde{P}_m f,$$

and show, by quoting a lemma based on the Brouwer fixed point theorem, or otherwise, that there is $v^* \in \tilde{H}_m$ such that $\Phi_m(v^*)=0$]

(c) (i) Use the above to show that there exists $u \in \tilde{V}$ with $\|u\| \leqslant R$ which satisfies the following equation

$$\nu\tilde{A}u + \tilde{B}(u, u) = f, \qquad (\ddagger *)$$

in $\tilde{V}'$.

(ii) Demonstrate the following regularity result: the solution, u, established above for equation $(\ddagger *)$ belongs to $D(\tilde{A})$ and that equation $(\ddagger *)$ holds in $\tilde{H}$.

END OF PAPER