UNIVERSITY OF CAMBRIDGE

MAMA/358, NST3AS/358, MAAS/358

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 9:00 am to 11:00 am

PAPER 358

SPECTRAL COMPUTATION IN INFINITE DIMENSIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $S \subset \mathbb{C}$ be a domain (non-empty connected open set) and $u : S \to \mathbb{R}$. We say that u is *subharmonic* if u is upper-semicontinuous and for any $z \in S$ and $r > 0$ such that $B_r(z) \subset S$,

$$u(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \, \mathrm{d}\theta.$$

Here, as in lectures, $B_r(z)$ denotes the closed ball of radius r and centre z. If $u : S \to \mathbb{R}$ is subharmonic and attains a maximum in S, prove that it must be constant.

Suppose that A is a closed and densely defined operator on a separable Hilbert space $\mathcal{H}$ (with norm $\|\cdot\|$) and $z_0 \notin \mathrm{Sp}(A)$. Show that there is a domain containing z_0 on which the function

$$z \mapsto \|(A - zI)^{-1}\|$$

is subharmonic. Deduce that if $\epsilon > 0$ and S is a bounded component of

$$\left\{ z \in \mathbb{C} : \left\|(A - zI)^{-1}\right\|^{-1} < \epsilon \right\},$$

then $S \cap \mathrm{Sp}(A) \neq \emptyset$.

(ii) Now suppose that $0 \notin \mathrm{Sp}(A)$. Let B be a bounded operator on $\mathcal{H}$ with $\|B\|\|A^{-1}\| < 1$. Prove that $0 \notin \mathrm{Sp}(A + B)$ and that

$$\|(A + B)^{-1}\| \leqslant \frac{\|A^{-1}\|}{1 - \|B\|\|A^{-1}\|}.$$

Let X be a bounded connected component of $\mathrm{Sp}(A)$ separated from the rest of the spectrum. Let B_n be a sequence of bounded operators with $\|B_n\| \to 0$ as $n \to \infty$. Show that given any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$\inf_{z \in \mathrm{Sp}(A + B_n)} \mathrm{dist}(z, X) < \epsilon.$$

(iii) Recall that an operator A on $\mathcal{H}$ is said to be compact if the closure of $\{Ax : \|x\| \leqslant 1\}$ is compact in $\mathcal{H}$. Show that any bounded, finite-rank operator is compact. Let A be compact and $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Define $A_n = \mathcal{P}_n^* \mathcal{P}_n A \mathcal{P}_n^* \mathcal{P}_n$, where $\mathcal{P}_n : \mathcal{H} \to \mathrm{span}\{e_1, \ldots, e_n\}$ is the corresponding orthogonal projection. Prove that $\|A - A_n\| \to 0$ as $n \to \infty$.

(iv) Let A be a (not necessarily self-adjoint) compact operator on a separable Hilbert space $\mathcal{H}$. Let $\{\mathcal{P}_n\}$ be a sequence of orthogonal projections onto finite-dimensional subspaces such that $\mathcal{P}_n$ converges strongly to the identity as $n \to \infty$. Prove that $\mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ converges to $\mathrm{Sp}(A)$ in the Hausdorff metric as $n \to \infty$.

(You may use the spectral theorem for compact operators, provided that it is clearly stated.)

2 Throughout this question, you may assume that for a closed and densely defined operator T on $\mathcal{H}$, T^*T is self-adjoint and its domain forms a core of T. You may also use the definition of $\mathrm{Sp}_{\mathrm{ess},k}$ and Sp_{d} given in lectures, together with their characterisation for normal operators.

(i) Let A be a closed and densely defined operator on a separable Hilbert space $\mathcal{H}$. Prove that

$$\begin{aligned}
\mathrm{Sp}_{\mathrm{ess},1}(A) &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI)) \cap \mathrm{Sp}_{\mathrm{ess}}((A - zI)(A - zI)^*)\}\,,\\
\mathrm{Sp}_{\mathrm{ess},2}(A) &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI))\}\\
\mathrm{Sp}_{\mathrm{ess},3}(A) &= \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{ess}}((A - zI)^*(A - zI)) \cup \mathrm{Sp}_{\mathrm{ess}}((A - zI)(A - zI)^*)\}\,.
\end{aligned}$$

Show that if A is normal, then

$$\mathrm{Sp}_{\mathrm{d}}(A) = \{z \in \mathbb{C} : 0 \in \mathrm{Sp}_{\mathrm{d}}((A - zI)^*(A - zI))\}\,.$$

Provide an example to show that this characterisation of the discrete spectrum need not hold if A is non-normal.

(ii) Let Ω_{N} denote the class of normal operators A on $l^2(\mathbb{N})$ for which the linear span of the canonical basis $\{e_j\}_{j=1}^{\infty}$ forms a core of A and A^*. For $A \in \Omega_{\mathrm{N}}$, the multiplicity of A at z is

$$h(z, A) = \begin{cases} \text{multiplicity of the eigenvalue } z, & \text{if } z \in \mathrm{Sp}_{\mathrm{d}}(A),\\ +\infty, & \text{if } z \in \mathrm{Sp}_{\mathrm{ess}}(A),\\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{P}_n$ be the orthogonal projection onto $\mathrm{span}\{e_1, \ldots, e_n\}$. List the singular values of $(A - zI)\mathcal{P}_n^*$ as $\sigma_n^{(n)}(z, A) \leqslant \sigma_{n-1}^{(n)}(z, A) \leqslant \cdots \leqslant \sigma_1^{(n)}(z, A)$. Prove that for any $j \in \mathbb{N} \cup \{0\}$, $\sigma_{n-j}^{(n)}(z, A)$ is decreasing in n and converges to a limit denoted by $\sigma_{\inf +j}(A - zI)$. Moreover, show that

$$h(z, A) = \lim_{k\to\infty} \sum_{j=0}^{k} \max\{0, 1 - k \times \sigma_{\inf +j}(A - zI)\}\,.$$

(You may assume standard properties of the Rayleigh–Ritz method.)

(iii) Let Λ be the evaluation of matrix entries with respect to the canonical basis, i.e., $A \mapsto \langle Ae_j, e_i\rangle$ for $i, j \in \mathbb{N}$. Prove that there exists a Δ_4^A-tower $\{h_{n_3,n_2,n_1}\}$ using Λ such that for any $\{z_{n_2,n_1}\}_{n_1,n_2\in\mathbb{N}} \subset \mathbb{C}$ with $\lim_{n_2\to\infty} \lim_{n_1\to\infty} z_{n_2,n_1} = z$,

$$\lim_{n_3\to\infty} \lim_{n_2\to\infty} \lim_{n_1\to\infty} h_{n_3,n_2,n_1}(z_{n_2,n_1}, A) = h(z, A) \quad \forall A \in \Omega_{\mathrm{N}}.$$

3

Let $\mathcal{C}(\mathcal{H})$ be the set of closed operators on a separable Hilbert space. For closed subspaces $M, N \subset \mathcal{H}$, we set

$$\delta(M,N) = \sup_{u \in M, \|u\|=1} \operatorname{dist}(u,N), \quad \hat{\delta}(M,N) = \max\{\delta(M,N), \delta(N,M)\}.$$

Here, $\delta(\{0\}, N) = 0$ and $\delta(M, \{0\}) = 1$ if $M \neq \{0\}$. Recall that the graph of an operator $A : \mathcal{D}(A) \to \mathcal{H}$ (where $\mathcal{D}(A)$ is a subspace of $\mathcal{H}$) is $\operatorname{gr}(A) = \{(x, Ax) : x \in \mathcal{D}(A)\}$. This is a subspace of the direct sum $\mathcal{H} \oplus \mathcal{H}$, and is closed precisely when A is a closed operator. Hence, given closed operators $S, T \in \mathcal{C}(\mathcal{H})$, we can define their gap as

$$\delta(S,T) = \delta(\operatorname{gr}(S), \operatorname{gr}(T)), \quad \hat{\delta}(S,T) = \max\{\delta(S,T), \delta(T,S)\}.$$

You may assume throughout that if S and T are densely defined, then $\hat{\delta}(S^*, T^*) = \hat{\delta}(S,T)$.

The class of nonlinear operator pencils Ω_{NL} is the set of maps $T : \mathbb{C} \mapsto \mathcal{C}(l^2(\mathbb{N}))$ satisfying the following two properties:

- T is continuous when $\mathcal{C}(l^2(\mathbb{N})$ is equipped with the topology induced by the gap $\hat{\delta}$;
- $\forall z \in \mathbb{C}$, $T(z)$ is densely defined and $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is a core of $T(z)$ and $T(z)^*$.

Let $T \in \Omega_{\mathrm{NL}}$. The spectrum and pseudospectra of T are

$$\operatorname{Sp}(T) = \{z \in \mathbb{C} : 0 \in \operatorname{Sp}(T(z))\}, \quad \operatorname{Sp}_\epsilon(T) = \operatorname{Cl}\left(\left\{z \in \mathbb{C} : \left\|T(z)^{-1}\right\|^{-1} < \epsilon\right\}\right).$$

Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of sets that satisfy the following:

- Each set G_n is a finite subset of $\mathbb{C}$;
- Any $z \in G_n$ is complex rational, i.e., $\operatorname{Re}(z) \in \mathbb{Q}$ and $\operatorname{Im}(z) \in \mathbb{Q}$;
- $\lim_{n \to \infty} \operatorname{dist}(z, G_n) = 0$ for any $z \in \mathbb{C}$.

For $T \in \Omega_{\mathrm{NL}}$, define the following set of evaluation functions:

$$\Lambda = \{T \mapsto \langle T(z)e_j, e_i \rangle : z \in G_n \text{ and } i,j,n \in \mathbb{N}\}.$$

Prove that

$$\{\operatorname{Sp}_\epsilon, \Omega_{\mathrm{NL}}, \mathcal{M}_{\mathrm{AW}}, \Lambda\} \in \Sigma_2^A, \quad \{\operatorname{Sp}, \Omega_{\mathrm{NL}}, \mathcal{M}_{\mathrm{AW}}, \Lambda\} \in \Pi_3^A,$$

where you may use the definitions of $\mathcal{M}_{\mathrm{AW}}$, Σ_j^A and Π_j^A given in lectures.

END OF PAPER