UNIVERSITY OF CAMBRIDGE

MAMA/357, NST3AS/357, MAAS/357

MAT3

MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 1:30 pm to 3:30 pm

PAPER 357

GRAVITATIONAL WAVES AND NUMERICAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In linearised theory, we consider spacetimes perturbatively close to Minkowski such that the metric in Cartesian coordinates $x^\alpha = (t,\, x,\, y,\, z)$ is given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}\,, \qquad \eta_{\alpha\beta} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)\,.$$

Here the metric perturbation is small, $h_{\alpha\beta} = \mathcal{O}(\epsilon)$ with $\epsilon \ll 1$, and we ignore terms quadratic or higher order in $h_{\alpha\beta}$. In the following you may use without proof that the Levi-Civita connection and Riemann tensor associated with a metric $g_{\alpha\beta}$ are

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} &= \frac{1}{2} g^{\alpha\rho}\left(\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma}\right)\,, \\ R^\gamma{}_{\rho\alpha\beta} &= \partial_\alpha \Gamma^\gamma_{\rho\beta} - \partial_\beta \Gamma^\gamma_{\rho\alpha} + \Gamma^\mu_{\rho\beta}\Gamma^\gamma_{\mu\alpha} - \Gamma^\mu_{\rho\alpha}\Gamma^\gamma_{\mu\beta}\,. \end{aligned}$$

(a) Writing the inverse metric as $g^{\alpha\beta} = \eta^{\alpha\beta} + k^{\alpha\beta}$, compute $k^{\alpha\beta}$ in terms of $h_{\alpha\beta}$.

(b) Compute the Levi-Civita connection $\Gamma^\alpha_{\beta\gamma}$ and the Riemann tensor $R_{\mu\rho\alpha\beta}$ to linear order in $h_{\alpha\beta}$. Show that the components $R_{\mu\nu\rho\sigma}$ are invariant under coordinate transformations $\tilde{x}^\alpha = x^\alpha - \xi^\alpha$, where the vector field ξ^α is small, $\xi^\alpha = \mathcal{O}(\epsilon)$.

(c) Consider the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with

$$\begin{aligned} h_{tt} &= h_{tx} \;=\; h_{ty} \;=\; h_{tz} \;=\; 0\,, \\ h_{xx} &= \left[\frac{x^2 - z^2}{r^2} - \frac{x^2(z^2 + 2y^2)}{r^4}\right] A + 2\frac{x^2(z^2 + 2y^2)}{r^4} B + \left[\frac{y^2 + 2z^2}{r^2} - \frac{x^2(z^2 + 2y^2)}{r^4}\right] C, \\ h_{xy} &= \frac{yx(x^2 - y^2)}{r^4}(A - 2B + C), \\ h_{xz} &= \frac{xz}{r^4}\left[(2x^2 + z^2)A + (-x^2 + 3y^2 + z^2)B - (x^2 + 2z^2 + 3y^2)C\right], \\ h_{yy} &= \left[\frac{z^2 - y^2}{r^2} + \frac{y^2(2x^2 + z^2)}{r^4}\right] A - 2\frac{y^2(2x^2 + z^2)}{r^4} B + \left[-\frac{x^2 + 2z^2}{r^2} + \frac{y^2(2x^2 + z^2)}{r^4}\right] C, \\ h_{yz} &= \frac{yz}{r^4}\left[-(z^2 + 2y^2)A - (3x^2 - y^2 + z^2)B + (3x^2 + 2z^2 + y^2)C\right], \\ h_{zz} &= \frac{y^4 - x^4}{r^4} A - 2\frac{z^2(x^2 - y^2)}{r^4} B + \frac{(x^2 - y^2)(r^2 + z^2)}{r^4} C. \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $A,\ B,\ C = \mathcal{O}(\epsilon)$ are perturbatively small functions of $t - r$.

(i) You may assume without proof that for this metric $\partial_m h_{im} = 0$. Briefly deduce that consequently $\partial^\nu h_{\mu\nu} = 0$. Show that the metric perturbation also satisfies $\eta^{\mu\nu} h_{\mu\nu} = 0$ and, hence, is in transverse-traceless gauge.

(ii) Consider an observer located in this spacetime at $x = y = 0$, $z \to \infty$. Compute the metric perturbation at the location of this observer and thus determine the gravitational-wave polarizations h_+ and $h_\times$ seen by this observer. Briefly interpret your results.

2

In the 3+1 formalism, the energy density, momentum density and stress tensor are defined in terms of the energy momentum tensor $T_{\mu\nu}$ by

$$\rho := T_{\mu\nu} n^\mu n^\nu \,, \qquad j_\alpha := -\perp^\mu{}_\alpha T_{\mu\nu} n^\nu \,, \qquad S_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta T_{\mu\nu} \,.$$

Here, n_μ is the timelike unit normal of the spacetime foliation and $\perp^\mu{}_\alpha = \delta^\mu{}_\alpha + n^\mu n_\alpha$ is the projector onto spatial hypersurfaces.

(a) Show that

$$T_{\alpha\beta} = \rho n_\alpha n_\beta + j_\alpha n_\beta + n_\alpha j_\beta + S_{\alpha\beta} \,.$$

(b) Show that the contracted spatial covariant derivative of the stress tensor can be written as

$$D_\mu S^\mu{}_\alpha := \perp^\rho{}_\mu \perp^\mu{}_\sigma \perp^\gamma{}_\alpha \nabla_\rho S^\sigma{}_\gamma = \perp^\gamma{}_\alpha \nabla_\mu S^\mu{}_\gamma - a_\sigma S^\sigma{}_\alpha \,,$$

where the acceleration vector a_σ is related to n_α in a form you should write down explicitly.

(c) Show that the Lie derivative of the momentum density along the unit timelike normal, $\mathcal{L}_{\boldsymbol{n}} j_\alpha := n^\mu \nabla_\mu j_\alpha + j_\mu \nabla_\alpha n^\mu$, satisfies

$$\perp^\alpha{}_\beta \mathcal{L}_{\boldsymbol{n}} j_\alpha = \mathcal{L}_{\boldsymbol{n}} j_\beta \,.$$

(d) Starting from the conservation of energy and momentum, $\nabla_\mu T^\mu{}_\alpha = 0$, show that the 3+1 evolution equation for the momentum density can be written as

$$\mathcal{L}_{\boldsymbol{n}} j_\alpha = c_1 D_\mu S^\mu{}_\alpha + c_2 S^\mu{}_\alpha a_\mu + c_3 K j_\alpha + c_4 \rho a_\alpha \,, \tag{†}$$

where $K := K^\mu{}_\mu$ is the trace of the extrinsic curvature $K_{\mu\nu}$ and c_1, c_2, c_3, c_4 are constants you should determine. [*Hint: You may use without proof that* $K_{\mu\alpha} = -\nabla_\mu n_\alpha - n_\mu a_\alpha$.]

(e) The timelike unit normal and the acceleration vector are related to the lapse function and shift vector by

$$n^\mu = \frac{1}{\alpha}(\partial_t - \boldsymbol{\beta})^\mu \,, \qquad a_\mu = \frac{\partial_\mu \alpha}{\alpha} \,.$$

Use these relations to substitute for the Lie derivative in Eq. (†) and write the 3+1 evolution equation for the momentum density in the form $\partial_t j_\mu = \ldots$ where you should determine and simplify as possible the right-hand side.

3

In Schwarzschild coordinates, the metric for a black hole spacetime of mass M is

$$\mathrm{d}s^2 = -\left(1-\tfrac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\tfrac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \quad t\in\mathbb{R},\ r\geqslant 0\,.$$

The maximal extension of this spacetime is obtained by transforming to Kruskal-Szekeres coordinates $\hat{t}$, $\hat{r}$ related to r and t by

$$\frac{\hat{t}}{\hat{r}} = \tanh\frac{t}{4M} \quad \text{for} \quad r > 2M\,, \qquad \frac{\hat{r}}{\hat{t}} = \tanh\frac{t}{4M} \quad \text{for} \quad r < 2M\,,$$
$$\hat{t}^2 - \hat{r}^2 = -e^{\frac{r}{2M}}(r-2M)\,.$$

(a) Sketch qualitatively in the plane spanned by $(\hat{t},\hat{r})$ the curves of constant t and those of constant r, including in particular the curves $r=2M$ and $r=0$.

(b) Consider an observer starting from rest, $\dot{\hat{r}}=0$, at $\hat{r}=0$, $\hat{t}=0$, freely falling into the black hole. Let n^μ denote the future pointing timelike unit normal of a foliation of a numerical simulation using geodesic gauge, i.e. lapse $\alpha=1$ and shift $\beta^i=0$, starting with initial data given by the hypersurface $\hat{t}=0$ of the Kruskal-Szekeres spacetime. Using the relation $n^\rho\nabla_\rho n_\mu = \partial_\mu\alpha$, show that the observer moves with four-velocity $u^\mu = n^\mu$. [*Hint: The Kruskal-Szekeres metric is diagonal.*]

(c) The geodesic motion of the observer in Schwarzschild coordinates obeys

$$-E^2 + \dot{r}^2 = -1 + \frac{2M}{r}\,,$$

where a dot denotes differentiation with respect to the observer's proper time τ and E is a constant. Show that the observer reaches $r=0$ at $\tau = M\pi$. Briefly comment on the implications of this result for the long-term stability of numerical simulations performed in the geodesic gauge. [*Hint:* $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\sqrt{x(1-x)} + \arcsin\sqrt{x}\right] = \sqrt{\frac{x}{1-x}}$]

(d) In the BSSN formulation of the Einstein equations, the time evolution of the trace K of the extrinsic curvature is given in terms of the inverse spatial metric γ^{mn}, the spatial covariant derivative D_m, the conformal traceless extrinsic curvature $\tilde{A}_{mn}$, the energy density ρ and the trace S of the stress tensor by

$$\partial_t K = \beta^m\partial_m K - \gamma^{mn}D_m D_n\alpha + \alpha\left(\tilde{A}_{mn}\tilde{A}^{mn} + \tfrac{1}{3}K^2\right) + 4\pi\alpha(\rho+S)\,.$$

(i) Show that for geodesic gauge, the energy condition $\rho + S \geqslant 0$ implies $\partial_t K \geqslant 0$.

(ii) Neglecting the term $\tilde{A}_{mn}\tilde{A}^{mn}$, assuming vacuum and using geodesic gauge, derive an ordinary differential equation for K. Solve this differential equation for the initial condition $K = K_0$ at $t = t_0$, where K_0 is a positive constant, and compute the time t_∞ when K becomes infinite.

END OF PAPER