UNIVERSITY OF CAMBRIDGE

MAMA/355, NST3AS/355, MAAS/355

MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 9:00 am to 12:00 pm

PAPER 355

BIOLOGICAL PHYSICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A reaction-diffusion model for the interaction of phytoplankton and zooplankton is

$$\frac{\partial u}{\partial t} = \nabla^2 u + u + u^2 - \gamma uv,$$
$$\frac{\partial v}{\partial t} = d\nabla^2 v + \beta uv - v^2,$$

where $\beta, \gamma > 0$. Find the regions in the $\beta - \gamma$ plane (a) in which there exists a homogeneous state (u^*, v^*) in which neither u^* nor v^* is zero, and in which it is stable to spatially uniform perturbations, and (b) in which that state may be unstable to a Turing instability. Find the critical wavenumber at the onset of the instability in terms of β and d. Show that the region in which there may be a Turing instability vanishes when $d = 1$.

2 Consider an elastic rod of length L, bending modulus A, mass per unit length λ, that is held vertically and clamped at the bottom end ($z = 0$). If the rod is only allowed to deflect transversely in the $x - z$ plane, show that small-amplitude deflections $X(z)$ in that plane under the action of gravity are described by the equation

$$AX_{zzzz} - (\sigma X_z)_z = 0,$$

where $\sigma(z)$ is the internal tension, which you should find. If the upper end is free, write down the complete set of boundary conditions on $X(z)$. Show that the function $u(z) = X_z(z)$ admits a similarity solution of the form

$$u = \eta^{1/3} F(\eta),$$

where

$$\eta = \frac{2}{3}\left[\lambda g \left(L - z\right)^3 / A\right]^{1/2}.$$

Noting that the differential equation for Bessel function J_ν has the form

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

show that $F = aJ_{-1/3} + bJ_{1/3}$, where a and b are undetermined constants, and you may use without proof the fact that J_α and $J_{-\alpha}$ are linearly independent for non-integer α. Find the associated boundary conditions on the function u, using the asymptotic form $J_\nu \sim x^\nu$ in the limit $x \to 0$. Find the critical condition for the rod to buckle under its own weight.

3 Bilayer lipid membranes interact with each other through attractive van der Waals interactions, screened electrostatic repulsion, and entropic repulsion, with interaction energies per unit area of membrane of the form

$$\begin{aligned} V_{\mathrm{vdW}} &= -\frac{A_H}{12\pi}\left(\frac{1}{d^2} - \frac{2}{(d+\delta)^2} + \frac{1}{(d+2\delta)^2}\right), \\ V_{\mathrm{elec}} &= Ae^{-d/\xi}, \\ V_{\mathrm{rep}} &= c\frac{(k_BT)^2}{k_c}\frac{1}{d^2}, \end{aligned}$$

where δ is the membrane thickness, d is the intermembrane spacing, A_H is the Hamaker constant, A is a constant, ξ is the Debye-Hückel screening length, c is a numerical constant, k_c is the membrane bending modulus, k_B is the Boltzmann constant and T is the absolute temperature.

(a) If the equilibrium membrane spacing d^* is found as the global minimum of the sum of these three terms, explain why, if there is a transition in d^* from some finite value to infinity as A_H constant is varied, then that transition is *discontinuous.*

(b) Now consider a stack of membranes with nearest-neighbour spacing d, with volume fraction $\phi = \delta/d$ in the limit $\delta \ll d$. In a mean field approximation as in the van der Waals approximation for gases, construct an effective free energy density $f(\phi)$, valid for $\phi \geqslant 0$, as the sum of two contributions, each of which is a power of ϕ: one from the entropic repulsion and the other from the combination of short-range electrostatic repulsion and the long-range van der Waals attraction, the latter expressed in terms of a second virial coefficient B_2 (whose general form should be stated, but which should not be calculated explicitly).

(c) From the general structure of B_2, show that there exists a critical value A_H^* below which $B_2 > 0$, and above which $B_2 < 0$ and that $B_2 \sim r(A_H^* - A_H)$ in the neighbourhood of A_H^* for some positive r which you need not find.

(d) Using the results in (c), deduce that there is a continuous "unbinding" transition as A_H is varied, such that the equilibrium spacing $d^* \sim (A_H - A_H^*)^{-\nu}$, where you should find the exponent ν.

END OF PAPER