UNIVERSITY OF CAMBRIDGE

MAMA/354, NST3AS/354, MAAS/354

MAT3

# MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 1:30 pm to 3:30 pm

## PAPER 354

## GAUGE/GRAVITY DUALITY

**Before you begin please read these instructions carefully**

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **TWO** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) Derive the unitary bound $\Delta \geqslant (d-2)/2$ for a scalar primary operator $\mathcal{O}$, in a dimension $d \geqslant 2$ CFT, excluding the case of the identity operator.

[*Hint: in Euclidean signature, the nonzero commutators in the conformal algebra are:*

$$\begin{aligned} [D, P_a] &= iP_a, & [D, K_a] &= -iK_a \\ [P_a, M_{bc}] &= i(\delta_{ab}P_c - \delta_{ac}P_b), & [K_a, M_{bc}] &= i(\delta_{ab}K_c - \delta_{ac}K_b) \\ [K_a, P_b] &= 2i(\delta_{ab}D - M_{ab}), \\ [M_{ad}, M_{bc}] &= i(\delta_{ab}M_{cd} - \delta_{ac}M_{bd} - \delta_{bd}M_{ca} + \delta_{cd}M_{ba}) \end{aligned}$$

*with $D$, $M_{ab}$, $P_a$, and $K_a$ being the generators of dilations, rotations, translations, and special conformal transformations respectively, in a convention where $D = i\Delta$ and $P_a^\dagger = -K_a$.*]

(b) A massless, minimally-coupled scalar field $\phi$ (with negligible interactions) lives in a 5+1 dimensional AdS-Poincaré spacetime, with metric:

$$ds^2 = \frac{1}{z^2}(dz^2 + dx^i dx^j \eta_{ij}),$$

where $\eta_{ij}$ is the Minkowski metric in $4+1$ dimensions, and the AdS radius has been set to unity.

(i) Write down the wave equation for $\phi$, and use it to determine the single permissible (conformally-invariant) boundary condition. What are the dimensions of the corresponding operator $\mathcal{O}$ and source $J$, in the dual holographic CFT$_5$?

[*Hint: you may wish to use the fact that* $\nabla^2 = \dfrac{1}{\sqrt{g}}\partial_a \sqrt{g} g^{ab} \partial_b$.]

(ii) Write down expressions for $\mathcal{O}$ and $J$ in terms of $\phi$, on the assumption that the source $J$ is a constant everywhere on the CFT boundary.

(iii) Now suppose that the source $J(x^i)$ is allowed to vary on the CFT boundary. Write down a new (more complicated) expression for $\mathcal{O}$ in terms of the $\phi$ field.

(c) Now consider the same CFT$_5$ on an Einstein universe $S^4 \times \mathbb{R}$, with the $\mathbb{R}$ direction being time, and the 4-sphere having radius $r$. Let the source $J = 0$. Using the operator-state correspondence, determine the allowed energy levels $E$ for CFT states whose bulk dual contains just *one* quantum of the $\phi$ field, and *no* angular momentum (in the low energy regime where interactions may be neglected).

**2**

The ABJM model is a large $N$ holographic CFT defined in 2+1 dimensions, in which the bulk Newton's constant scales like $G \sim N^{-3/2}$, for unit AdS radius. Let this CFT be in the vacuum state of 2+1 Minkowski spacetime. For this problem you will consider only the $t = 0$ Cauchy slice $\Sigma$ of the boundary theory.

Consider two disjoint disk regions $D_1$ and $D_2$ defined on $\Sigma$, of radius $R_1$ and $R_2$ respectively. Let $L > 0$ be the shortest distance between points on the boundaries of the disks, so that $R_1 + L + R_2$ is the distance between their centres. The mutual information is defined as:

$$I = S(D_1) + S(D_2) - S(D_1 \cup D_2),$$

where $S(A)$ refers to the entanglement entropy of a region $A$. Let $I_c$ be the contribution to $I$ which is at leading order in a $1/N$ expansion.

(a) Write down the holographic entropy formula for $S(D_1)$, the entropy of a single disk at leading order in $N$. Derive a formula showing the functional dependence of $S(D_1)$ on $\epsilon$, the UV cutoff on the holographic coordinate $z$. [You do not need to determine the numerical coefficient of the finite term, but you do need to determine the coefficients of any divergent terms, in terms of $G$.]

(b) (i) Using holographic entropy, prove that $I_c \geqslant 0$.

(ii) Briefly argue that, for a given $R_1$ and $R_2$, there should exist a critical radius $L = C(R_1, R_2)$ such that:

- when $L < C$, $I_c > 0$, and
- when $L > C$, $I_c = 0$.

[In this part you do not need to determine the value of $C$.]

(iii) Without calculating it, sketch roughly how you expect $S(D_1 \cup D_2)$ to behave as a function of $L$, in the vicinity of the phase transition at $L \approx C$. ($R_1$ and $R_2$ are still fixed.)

(c) Calculate the dependence of $C$ on $R_1$ and $R_2$. That is, calculate $C(R_1, R_2)$ up to a single undetermined parameter $X$. *[Hint: use conformal symmetry.]*

## END OF PAPER