UNIVERSITY OF CAMBRIDGE

MAMA/352, NST3AS/352, MAAS/352

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 1:30 pm to 3:30 pm

PAPER 352

NON-NEWTONIAN FLUID MECHANICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) A *small-amplitude oscillatory rheometer* imposes a strain $\gamma = \epsilon \sin \omega t$ on a viscoelastic fluid sample, with $\epsilon \ll 1$, and measures the shear-stress response τ, which can be decomposed into the form $\tau = \epsilon\, a\, \sin(\omega t + \delta)$, where the magnitude $a(\omega)$ and phase shift $\delta(\omega)$ are measured. How do a and δ relate to the storage and loss moduli, G' and G'', of the fluid? What would you expect to observe in the two limits $\delta(\omega \to 0)$ and $\delta(\omega \to \infty)$ for a 'standard' viscoelastic fluid? Explain how to use measurements of $\delta(\omega)$ to estimate the characteristic elastic relaxation time of the fluid.

(b) A *parallel-plate rheometer* consists of two parallel disks of radius R, separated by a thin gap of depth h, filled with a viscous fluid. The lower disk is stationary, while the upper disk is rotated at a rotation rate $\Omega > 0$, which, in a steady state, requires an imposed torque T. The relationship between T and Ω provides information about the fluid's rheology.

Write down the expected strain rate $\dot\gamma$ for the fluid in the parallel-plate rheometer, and an integral expression for the torque T in terms of the shear stress in the fluid. Assuming that the fluid can be described by a generalised Newtonian constitutive relationship with viscosity $\eta(\dot\gamma)$, show that the torque can be converted into an integral over the strain rate of the form

$$T = \int_0^{\dot\gamma_R} \eta\, f(\dot\gamma, R, \dot\gamma_R)\, d\dot\gamma, \qquad (*)$$

for some function f, where $\dot\gamma_R$ is a reference value of strain rate that you should define.

Use $(*)$ to obtain an expression for the viscosity, and thus show how one can use measurements of T as a function of Ω from the parallel-plate rheometer to determine $\eta(\dot\gamma)$ for the fluid.

An experiment is performed in which Ω and T are found to be related by $\Omega = \alpha h(T - T_c)/R$ for some $\alpha > 0$ and $T_c > 0$. Sketch $\eta(\dot\gamma)$ in this case.

2

A rheological model for the deviatoric stress $\boldsymbol{\tau}$ of a certain polymeric solution can be written in terms of a symmetric structure tensor $\boldsymbol{\alpha}$ and strain rate tensor $\dot{\boldsymbol{\gamma}} = \boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T$ as

$$\boldsymbol{\tau} = \eta_0 \dot{\boldsymbol{\gamma}} + b_1 \boldsymbol{\alpha} + b_2 \left[\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha} \cdot \dot{\boldsymbol{\gamma}} - \frac{2}{3} \left(\boldsymbol{\alpha} : \dot{\boldsymbol{\gamma}} \right) \boldsymbol{I} \right],$$

where $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{\alpha}$ obeys the evolution equation

$$\overset{\triangledown}{\boldsymbol{\alpha}} + \frac{\xi}{2} \left(\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha} \cdot \dot{\boldsymbol{\gamma}} \right) + c_1 \boldsymbol{\alpha} = c_2 \dot{\boldsymbol{\gamma}},$$

and η_0, b_1, b_2, ξ, c_1 and c_2 are all non-negative constants. Here $\overset{\triangledown}{\boldsymbol{\alpha}}$ represents the upper convected derivative.

(a) Show that this model reduces to the usual Oldroyd-B model in some limit, which you should specify. Under what conditions is the Oldroyd-A model recovered?

(b) Consider now the case $\xi = 1$, with the remaining parameters unspecified. Calculate the shear viscosity and the two normal stress differences predicted by this model in steady simple shear. Under what conditions does the model predict shear thinning for the viscosity?

(c) Again with $\xi = 1$, consider the behaviour of this polymeric solution under steady uni-axial extension, with velocity given in Cartesian coordinates by $\boldsymbol{u} = \dot{\epsilon} \left(-x/2, -y/2, z \right)$ for a constant extensional strain rate $\dot{\epsilon}$.

Show that the extensional viscosity η_{ext} is a constant when $b_2 = 0$, and find a general expression for η_{ext} when $b_2 \neq 0$.

3

(a) The so-called *slump test* provides a simple method to estimate a fluid's yield stress. A cylindrical cup filled with a volume V of a yield-stress fluid of density ρ is upturned on a horizontal surface, and then removed. The fluid slumps and spreads axisymmetrically under gravity, coming to rest as a deposit with radius R.

You should assume that the yield stress is sufficiently weak that all the deposited fluid flows initially when the cup is removed. You should further assume that the flow is well described by lubrication theory and neglect any effects of surface tension. Explain how measurement of R can provide an estimate of the fluid's yield stress. Derive an expression for this estimate.

Suppose the test is performed using dry sand rather than a fluid, and again assume that all the sand flows when the cup is removed. What shape would you expect the final deposit to take in this case? Briefly explain your answer.

(b) An alternative test is proposed using two large vertical plates, located at $x = -l$ and $x = l$, creating a slot between them with an open top and bottom. The slot is filled with a yield-stress fluid of density ρ; if the fluid is immobile, the test is reset with a slightly wider half-width l and performed again. The critical half-width $l = l_c$ at which the fluid first flows down the slot is recorded.

Again neglecting any effects of surface tension, explain how measurement of l_c provides an estimate of the fluid's yield stress and derive an expression for this estimate.

Suppose that the fluid is described by a Herschel–Bulkley rheology, which, in one dimension, relates the shear stress τ and strain rate $\dot{\gamma}$ by $\tau = \operatorname{sgn}\{\dot{\gamma}\}\,(\tau_y + K\,|\dot{\gamma}|^n)$ when $|\tau| > \tau_y$, with $\dot{\gamma} = 0$ otherwise, where $\tau_y \geqslant 0$ is the yield stress, $K > 0$ is the consistency and $n > 0$ is the power-law index. Find the flux Q of fluid down the slot when $l > l_c$.

A test is performed on a fluid with an unknown rheology in which it is observed that $Q \sim (l - l_c)^2$ for $l - l_c \ll l_c$ and that $Q \sim l^4$ when $l \gg l_c$. Sketch a plausible constitutive law $\tau(\dot{\gamma})$ for this fluid, noting any relevant scalings.

END OF PAPER