UNIVERSITY OF CAMBRIDGE

MAMA/346, NST3AS/346, MAAS/346

MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 9:00 am to 12:00 pm

PAPER 346

FORMATION OF GALAXIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Jeans equation reads

$$\frac{\partial}{\partial t}\left(\rho\langle v_j\rangle\right) + \frac{\partial}{\partial x_i}\left(\rho\langle v_i v_j\rangle\right) + \rho\frac{\partial \phi}{\partial x_j} = 0,$$

where x_i and v_i are Cartesian position and velocity components, ρ is the density and ϕ the potential, whilst angled brackets denote averages over the distribution function. From this starting point, derive the tensor virial theorem in the form

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2T_{jk} + \Pi_{jk} + W_{jk}.$$

Here, I_{jk} is the moment of inertia tensor, T_{jk} is the kinetic energy of ordered motion, Π_{jk} is the kinetic energy of random motion and W_{jk} is the potential energy tensor.

Consider a galaxy of mass M that is an oblate spheroid and is in a steady state. Let us assume that the only streaming motions are around the symmetry axis or z-axis. Let us define an orbital anisotropy δ using the equation $\Pi_{zz} = (1-\delta)\Pi_{xx}$. From the tensor virial theorem, show that

$$\frac{v_0^2}{\sigma_0^2} = (1-\delta)\frac{W_{xx}}{W_{zz}} - 1,$$

where $T_{xx} = Mv_0^2/2$ and $\Pi_{xx} = M\sigma_0^2$.

If the equidensity surfaces are stratified on similar concentric spheroids, the ratio W_{xx}/W_{zz} depends only on the ellipticity ϵ and is well approximated by $W_{xx}/W_{zz} \approx (1-\epsilon)^{-0.9}$. On a plot of ϵ versus v_0/σ_0, draw lines corresponding to an oblate spheroidal galaxy viewed edge-on with different velocity anisotropies $\delta = 0, 0.1, 0.2, 0.3$.

Describe qualitatively how the curves change when the galaxy is viewed at an arbitrary inclination angle.

Mark on the plot the domain of high mass elliptical galaxies and of low mass elliptical galaxies.

Using this plot, describe evolutionary pathways that lead to high mass elliptical galaxies and to low mass elliptical galaxies.

Give four observational differences between high mass and low mass elliptical galaxies.

2 Explain why the distribution function of a galaxy obeys the collisionless Boltzmann equation.

If F is the distribution function, show that the collisionless Boltzmann equation in spherical polar coordinates (r, θ, ϕ) is

$$\frac{\partial F}{\partial t} + \dot{r}\frac{\partial F}{\partial r} + \dot{\theta}\frac{\partial F}{\partial \theta} + \dot{\phi}\frac{\partial F}{\partial \phi} + \dot{v}_r\frac{\partial F}{\partial v_r} + \dot{v}_\theta\frac{\partial F}{\partial v_\theta} + \dot{v}_\phi\frac{\partial F}{\partial v_\phi} = 0,$$

Here, (v_r, v_θ, v_ϕ) are velocity components referred to the spherical polar coordinate system.

Show that

$$\dot{r} = v_r, \qquad \dot{\theta} = v_\theta/r, \qquad \dot{\phi} = v_\phi/(r\sin\theta).$$

By using Lagrange's equations, show that the components of the acceleration are

$$\dot{v}_r = \frac{v_\theta^2 + v_\phi^2}{r} - \frac{d\phi}{dr}, \qquad \dot{v}_\theta = \frac{v_\phi^2\cot\theta - v_r v_\theta}{r}, \qquad \dot{v}_\phi = \frac{-v_\phi v_r - v_\phi v_\theta \cot\theta}{r},$$

where the potential is assumed to be spherically symmetric or $\phi = \phi(r)$.

Let us consider spherically symmetric infall onto a spherical cluster with density $\rho(r, t)$. We denote averages over the distribution function by angled brackets. As the infall is spherical, the mean velocities $\langle v_\theta \rangle$ and $\langle v_\phi \rangle$ both vanish. Show that the continuity equation has the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\langle v_r \rangle}{\partial r} + \frac{2}{r}\rho\langle v_r \rangle = 0.$$

Let us assume that the velocity distributions have isotropic dispersions about the mean, so the second velocity moments are

$$\langle v_r^2 \rangle = \sigma^2 + \langle v_r \rangle^2, \qquad \langle v_\theta^2 \rangle = \sigma^2, \qquad \langle v_\phi^2 \rangle = \sigma^2.$$

Show that the Jeans equation takes the form

$$\frac{\partial}{\partial r}\left(\rho\sigma^2\right) + \rho\frac{\partial \langle v_r \rangle}{\partial t} + \rho\langle v_r \rangle\frac{\partial \langle v_r \rangle}{\partial r} = -\frac{d\phi}{dr}.$$

Assuming a ΛCDM cosmology, explain why

$$\frac{d\phi}{dr} = \frac{GM(r)}{r^2} + \frac{4\pi G}{3}\rho_{\rm b} r - \frac{\Lambda}{3}r,$$

where $\rho_{\rm b}$ is the homogeneous cosmological background density and Λ is the cosmological constant.

If a is the scale factor, we define the acceleration parameter q as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda,$$

where

$$\Omega_{\rm m} = \frac{8\pi G\rho_{\rm b}}{3H^2}, \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2},$$

and H is the Hubble parameter. Show that

$$\frac{d\phi}{dr} = \frac{GM(r)}{r^2} + qH^2 r.$$

Now suppose the radial velocity can be written as the sum of the Hubble flow and a peculiar infall velocity

$$\langle v_r \rangle (r,t) = H(t) r + v_{\mathrm{p}}(r,t).$$

Show that the Jeans equation can be recast as

$$\frac{\partial}{\partial r}\left(\rho\sigma^2\right) = -\rho\left[\frac{GM(r)}{r^2} + S(r,t)\right],$$

where

$$S(r,t) = v_{\mathrm{p}}\frac{\partial v_{\mathrm{p}}}{\partial r} + H\left(v_{\mathrm{p}} + r\frac{\partial v_{\mathrm{p}}}{\partial r}\right) + \frac{\partial v_{\mathrm{p}}}{\partial t}.$$

3 (a) Consider an Einstein-de Sitter Universe that consists of just baryons and photons. Show that, prior to recombination, the photon/baryon fluid has sound speed

$$c_{\mathrm{s}} = \frac{c}{\sqrt{3}}\left[1 + \frac{3\rho_{\mathrm{b}}}{4\rho_{\mathrm{r}}}\right]^{-1/2},$$

where ρ_{b} and ρ_{r} are the density of baryons and radiation.

If $a(t)$ is the scale factor, show that the behaviour of the sound speed before and after matter-radiation equality t_{eq} behaves like

$$c_{\mathrm{s}} \propto \begin{cases} a^0 & \text{if } t < t_{\mathrm{eq}}, \\ a^{-1/2} & \text{if } t > t_{\mathrm{eq}}. \end{cases}$$

How does the sound speed behave after recombination?

The comoving Jeans length of a collisional fluid with homogenous density $\bar{\rho}$ and sound speed c_{s} takes the form

$$\lambda_{\mathrm{com}} = \frac{c_{\mathrm{s}}}{a}\sqrt{\frac{\pi}{G\bar{\rho}}}.$$

Deduce how the comoving Jeans length λ_{com} for the baryons behaves from earliest times to after the epoch of recombination t_{rec}.

On a graph of comoving scale versus time, show the evolution of the comoving Jeans length for adiabatic, isentropic perturbations of baryons.

Now suppose the Universe also contains dark matter particles which can be treated as a collisionless fluid. How does the formula for the comoving Jeans length change?

Assuming the dark matter is a cold relic, derive the behaviour of the comoving Jeans length for cold dark matter particles, identifying four epochs in which the behaviour is different.

On a new plot, show the behaviour of λ_{com} for cold relics from earliest times to after the epoch of recombination t_{rec}.

(b) The Local Group contains two massive galaxies, the Milky Way and M31, that are falling towards each other on a radial orbit. If r is the separation and M is the total mass of the galaxies, show that

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{1}{2}\frac{GM}{a},$$

where $v = \dot{r}$ and $2a$ is the maximum value of r achieved on the orbit.

Use the substitution $r = a(1 - \cos 2\eta)$ to demonstrate

$$2\eta - \sin 2\eta = \left(\frac{Gm}{a^3}\right)^{1/2} t,$$

where t is the age of the universe.

Defining $\Omega \equiv (GM/r^3)^{1/2}$, show that

$$\Omega t = \frac{\eta - \sin\eta\cos\eta}{2^{1/2}\sin^3\eta}.$$

Defining $\omega = v/r$, also show that

$$\omega t = \frac{(\eta - \sin\eta\cos\eta)\cos\eta}{\sin^3\eta}.$$

By eliminating η from the last two equations, we can obtain Ω as a function of ω. It is found numerically that an excellent approximation to the solution is given by

$$\Omega t + 0.85\omega t = 2^{-3/2}\pi.$$

Now consider a very remote small dwarf galaxy also belonging to the Local Group with separation r_i from the mass centre. It may be assumed the dwarf galaxy is much further away from the mass centre than either the Milky Way or M31. Explain why

$$\Omega_i t + 0.85\omega_i t = 2^{-3/2}\pi,$$

where $\Omega_i = (GM/r_i^3)^{1/3}$ and $\omega_i = v_i/r_i$.

Suppose r_0 is the distance (measured from the mass centre) at which the expansion was stopped by the gravity of the Local Group. Show that

$$r_0 = r_i\left[1 - \frac{0.85}{2^{-3/2}\pi}\omega_i t\right]^{2/3}.$$

4 Let us assume an Einstein-de Sitter Universe with zero cosmological constant and zero curvature. Suppose $M_{\rm i}$ is the mass inside $r_{\rm i}$ at time $t_{\rm i}$. Show that

$$M_{\rm i} = \frac{2{r_{\rm i}}^3}{9G{t_{\rm i}}^2},$$

where $t_{\rm i}$ is in the matter dominated epoch.

Now let us consider the effects of a spherically symmetric overdensity $\delta M_{\rm i}(r_{\rm i})$ so that

$$M_{\rm i} = \frac{2{r_{\rm i}}^3}{9G{t_{\rm i}}^2} + \delta M_{\rm i}(r_{\rm i}).$$

Explain why the dark matter shell at $r_{\rm i}$ has initially a radial velocity $v_{\rm i}$ given by

$$v_{\rm i}(r_{\rm i}) = \frac{2r_{\rm i}}{3t_{\rm i}}.$$

Show that the position $r(r_{\rm i}, t)$ of each shell at time t is given by

$$\frac{d^2r}{dt^2} = -\frac{GM(r,t)}{r^2}, \qquad M(r,t) = \int_0^\infty dr_{\rm i}\frac{dM}{dr_{\rm i}}H(r - r(r_{\rm i},t)),$$

where $H(x)$ is the Heaviside function.

The radius $r(r_{\rm i}, t)$ of a given shell initially increases till it reaches a maximum value or turnaround radius $r_\star(r_{\rm i})$ at a turnaround time $t_\star(r_{\rm i})$. Show that

$$t_\star(r_{\rm i}) = \frac{\pi}{2}\sqrt{\frac{r_\star^3}{2GM_{\rm i}(r_{\rm i})}}, \qquad r_\star(r_{\rm i}) = r_{\rm i}\frac{M_{\rm i}(r_{\rm i})}{\delta M_{\rm i}(r_{\rm i})}.$$

Describe qualitatively the motion of the shells at times $t > t_\star$.

We will now derive an equation for the evolution of shells after turnaround. Let us assume that the solutions have the properties of self-similarity. A solution is self-similar if it remains identical to itself after all distances have been rescaled by a time-dependent length $R(t)$ and all masses by a time-dependent mass $M(t)$. Here, $R(t)$ is taken to be the radius at which dark matter particles are turning around at time t, while $M(t)$ is taken to be the mass interior to $R(t)$ at time t. In other words, $R(t) = r_\star(r_{\rm i})$ and $M(t) = M_i(r_i)$ such that $t_\star(r_{\rm i}) = t$. You may assume that the self-similar solutions have the form

$$M(r,t) = M(t)\mathcal{M}(r/R(t)), \qquad r(r_{\rm i},t) = r_\star(r_{\rm i})\Lambda(t/t_\star(r_{\rm i})),$$

where $\mathcal{M}$ and Λ are functions of a single variable. Show that

$$\frac{d^2\Lambda}{d\tau^2} = -\frac{\pi^2}{8\Lambda^2}\frac{M(t)}{M_{\rm i}(r_{\rm i}}\mathcal{M}\left(\frac{r_\star(r_{\rm i})}{R(t)}\right).$$

Now assume that the initial conditions are

$$\frac{\delta M_{\rm i}(r_{\rm i})}{M_{\rm i}(r_{\rm i})} = \left(\frac{M_0}{M_{\rm i}(r_{\rm i})}\right)^\epsilon,$$

where M_0 and ϵ are parameters with $0 \leqslant \epsilon \leqslant 1$. What do the cases $\epsilon = 0$ and $\epsilon = 1$ correspond to physically?

Show that

$$t_\star = \frac{3\pi}{4} t_{\rm i} \left(\frac{M_{\rm i}}{M_0}\right)^{3\epsilon/2}, \qquad r_\star(M_{\rm i}) = \left[\frac{8}{\pi^2} t_\star^2(M_{\rm i}) G M_{\rm i}\right]^{1/3}.$$

Hence, show that

$$M(t) = M_0 \left(\frac{4t}{3\pi t_{\rm i}}\right)^{2/(3\epsilon)}, \qquad R(t) = \left[\frac{8t^2 G}{\pi^2} M(t)\right]^{1/3}.$$

Deduce that

$$\frac{M(t)}{M_{\rm i}} = \tau^{2/(3\epsilon)} \qquad \frac{r_\star(M_{\rm i})}{R(t)} = \tau^{-2/3-2/(9\epsilon)}.$$

Finally, show that the equation of motion for the shells after $t > t_\star$ is

$$\frac{d^2\Lambda}{d^2\tau} = -\frac{\pi^2}{8} \frac{\tau^{2\epsilon/3}}{\Lambda^2} \mathcal{M}\left(\frac{\Lambda}{\tau^{2/3+2/(9\epsilon)}}\right).$$

(This equation does not have an analytic solution but can be integrated numerically to give the evolution of the shells after turnaround).

END OF PAPER