UNIVERSITY OF CAMBRIDGE

MAMA/345, NST3AS/345, MAAS/345

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 1:30 pm to 4:30 pm

PAPER 345

FLUID DYNAMICS OF THE ENVIRONMENT

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an incompressible, inviscid, non-diffusive Boussinesq fluid with a linearly stratified background density, denoted by $\hat{\rho}(z)$. Let x and z be the horizontal and vertical coordinates, respectively, with z directed upward, and let t denote time. A thin cylinder, whose axis is perpendicular to the x–z plane, undergoes small-amplitude oscillations along the line $x = 0$, generating two-dimensional motion.

(a) Starting from equations for the conservation of volume, mass and momentum, show that for small perturbations

$$\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} + N^2 \frac{\partial^2}{\partial x^2}\right] w = 0,$$

where w is the vertical fluid velocity and N is the buoyancy frequency that you should define carefully. You should carefully define all the parameters that you introduce.

(b) Sketch and describe the wave pattern that develops around the oscillating cylinder for the oscillation frequency ω. How does the wave pattern change for different values of ω? What is the geometric relationship between the group velocity, the phase velocity and the wave vector?

(c) Now suppose that the topography of the bottom is given by

$$H(x) = \begin{cases} \frac{3}{8} & x \leqslant -\frac{\sqrt{3}}{4} \\ 2x^2 & -\frac{\sqrt{3}}{4} \leqslant x \leqslant \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leqslant x \end{cases}$$

and the cylinder oscillates around the point $(0, s)$, where $s > 0$, with the oscillation frequency ω such that the angle of the excited internal gravity wave is at most $\theta = \frac{\pi}{6}$ to the vertical. Consider the initial ray that is directed into the positive x and negative z direction. Using ray tracing, show that there is a range of values of θ and s such that the reflected ray (after potentially multiple reflections) is directed into the second quadrant (i.e., into the negative x and positive z direction).

2 Consider a fluid layer of depth $h(x,t)$ and density $\rho_1(x,t)$, flowing with horizontal velocity $u(x,t)$ in a channel of constant width b with a flat bottom located at $z=0$. The horizontal coordinate x is aligned with the channel, and the vertical coordinate z points upward; t denotes time. In the region $z > h(x,t)$ lies an infinite quiescent layer of ambient fluid with constant density $\rho_2 < \rho_1$. Across the interface at $z = h(x,t)$, the fluid layer exchanges mass with the ambient fluid: it entrains ambient fluid at a constant velocity w_e and detrains into the ambient fluid at a constant velocity w_d.

(a) Considering a small volume inside the fluid layer, set up a linearised set of shallow-water equations for the conservation of volume, mass and momentum. You should state clearly any assumptions that you make.

(b) Now assume that the density difference between ρ_1 and ρ_2 is small. Show that the set of governing shallow-water equations can be written in the matrix form:

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u & h \\ \frac{h}{2} & g' & u \end{pmatrix} \begin{pmatrix} g' \\ h \\ u \end{pmatrix}_x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g' \\ h \\ u \end{pmatrix}_t = \begin{pmatrix} -g' \frac{w_e}{h} \\ w_e - w_d \\ -u \frac{w_e}{h} \end{pmatrix} \tag{1}$$

where g' is the reduced gravity that you should define carefully.

(c) Is the system hyperbolic? Find characteristics of the system and determine equations along each of the characteristics. What happens in the limit $w_e, w_d \to 0$?

3 Consider a well-insulated, long corridor—such as in a data centre—with height H and uniform width W. At one end of the corridor is a very large sliding door that spans the entire cross-section of the corridor. The door is opened at regular intervals for a duration δt, with a frequency of n times per hour. Each time the door opens, a cold gravity current enters along the floor, enabling an exchange of air with the colder outdoor environment. This process removes heat generated by equipement within the corridor.

The air inside the corridor is assumed to be well mixed, except during the periods when the door is open. The door is triggered to open whenever the indoor temperature reaches a prescribed maximum value T. The outdoor ambient remains at a constant temperature T_0, with $T > T_0$. The density of outdoor air at T_0 is ρ_0, while the density of the indoor air at temperature T is $\rho = \rho_0 - \hat{\rho}$.

(a) Consider the Navier–Stokes equations for a Boussinesq fluid,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\mathbf{g} + \nu \nabla^2 \mathbf{u},$$

where the density is expressed as $\rho = \rho_0 + \rho'$, with ρ_0 as the reference density and ρ' representing a small perturbation. The kinematic viscosity is denoted by ν, and $\mathbf{g}$ is the gravitational acceleration. Derive the vorticity equation, explicitly identifying the baroclinic generation term.

(b) To derive the expression for the gravity current (see Figure 1 on the next page), write down the steady, inviscid, two-dimensional vorticity equation. Consider a control volume surrounding the gravity current in a frame moving with the front. Let the velocity upstream of the current be u_1 and the velocity downstream be u_2. Assuming a sharp density interface (i.e., neglecting entrainment) and using a suitable velocity profile (or otherwise), evaluate the vorticity generation within the control volume and the advection of vorticity across its boundaries. Hence, derive an expression for the downstream velocity u_2 in the frame of the gravity current, and show that

$$u_2 = \sqrt{2g'h},$$

where h is the depth of the current and $g' = \frac{\hat{\rho}}{\rho}g$. Also derive an expression for the front velocity in the lab frame.

(c) Give physical reason why you might expect the gravity current to move with a constant speed.

(d) Assuming the flow is energy-conserving, sketch the gravity current, state the relationship between the depth of the current, h, and H, and derive an expression for the exchange flow during the time interval δt while the door remains open.

(e) Apply an energy balance and determine the maximum indoor temperature T, if heat is produced at a rate $\dot{q}$ (J/s).

(f) Why it is desirable to stop mixing when the door is opened to allow exchange?

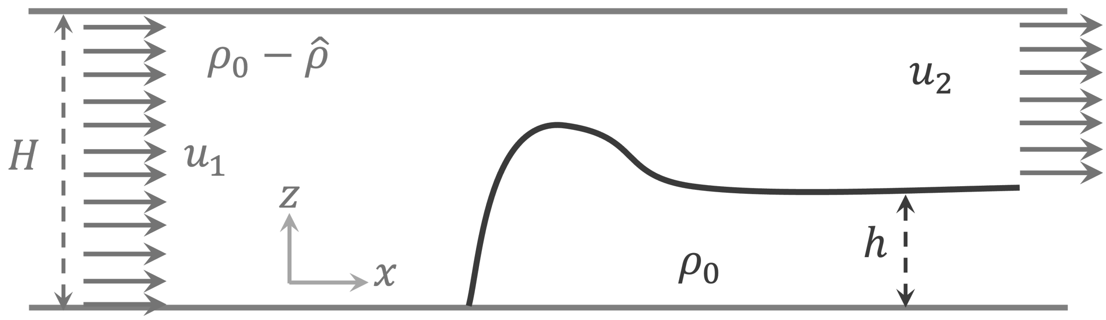

Figure 1: Schematic of the side view of a gravity current in the frame of the front.

4 Consider a vertical wall, $x = 0$ that supplies a constant heat flux,

$$-K_T \frac{\partial T}{\partial x} = q_0,$$

to an infinite ambient fluid (air) occupying the region $x > 0$ and maintained at a uniform temperature T_0. At the base of the wall, a line source introduces a buoyancy flux B_0, initiating a two-dimensional plume. This plume, which initially carries both volume flux, Q_0 and momentum flux M_0, entrains ambient fluid and gains heat as it rises due to the wall flux. The initial plume width b_0, along with its volume and momentum fluxes at $z = 0$, are chosen to permit a self-similar description of the flow.

We define (each per unit length of the line source)

Momentum flux per unit mass: $M = b\,W^2$

Volume flux: $Q = b\,W$

Temperature, density coupling: $\dfrac{T - T_0}{T_0} = \dfrac{(\rho_0 - \rho)}{\rho_0}$

Buoyancy flux: $\mathcal{B} = b\,W\,\dfrac{(\rho_0 - \rho)}{\rho_0}\,g \equiv b\,W\,\dfrac{(T - T_0)}{T_0}\,g$

where z measures the vertical height, $b = b(z)$ is the width of the plume, $W = W(z)$ is the tophat vertical velocity, $T = T(z)$ is the tophat temperature of the plume fluid.

(a) Explain the Boussinesq approximation and Batchelor's entrainment hypothesis. Discuss how each applies in the context of this plume.

(b) Write down the advection–diffusion equation for temperature. Scale the equation using characteristic height h_0, width b_0, and vertical velocity w_0. Use continuity to estimate the spanwise velocity scale. Assuming a slender plume, i.e. $\frac{b_0}{h_0} = \epsilon \ll 1$, show that vertical diffusion is negligible compared to horizontal diffusion.

(c) Integrate the volume, momentum, and advection–diffusion equations across the plume cross-section (assuming top-hat profiles, and and applying entrainment assumption). Combine the integrated advection–diffusion and volume flux equations to derive an evolution equation for the buoyancy flux $\mathcal{B}(z)$.

(d) Perform a dimensional analysis of the governing equations and show the scaling relations, $Q \sim z^{4/3}$ and $M \sim z^{5/3}$.

(e) Introduce a suitable substitution of variable and subsequently similarity variables and obtain the similarity solution for the plume quantities; momentum flux, volume flux, width and temperature of the plume.

(f) Using the similarity solution, determine the expression for the initial plume width b_0.

END OF PAPER