UNIVERSITY OF CAMBRIDGE

MAMA/343, NST3AS/343, MAAS/343

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 1:30 pm to 3:30 pm

PAPER 343

QUANTUM ENTANGLEMENT IN MANY-BODY PHYSICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a. Given a quantum system described by a density matrix, then the most general type of time evolution is given by the Lindblad equation. Write down the most general form of such a Lindblad equation. Are there any properties that the Lindblad operators have to satisfy?

b. What are the conditions on the Lindblad operators such that this Lindblad equation converges to a unique fixed point in the long-time limit?

c. Write down a specific Lindblad equation for a 2-qubit system such that, in the long time limit, it converges to the state $\rho(\infty) = |\psi^-\rangle\langle\psi^-|$ with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt(2)$. What is the smallest (in magnitude) non-zero eigenvalue of this Lindbladian ("the gap")?

d. How does the "unravelling" of this time-evolved state look like in terms of a pure state or, in other words, can you write down the purification of this Lindblad equation? What happens if the fixed point of the Lindblad equation is a pure state as in the previous example?

2

a. What is the monogamy property of entanglement? Can you give an example for the case of 3 qubits?

b. Given a system of $N \to \infty$ qubits and the Heisenberg Hamiltonian

$$\hat{H} = \sum_{ij}^{N} X_i X_j + Y_i Y_j + Z_i Z_j$$

where the sum over i and j runs over all sites and thus the Heisenberg term acts over any pair of qubits, determine the ground state energy. What is the difference between the energy of the ground state and the energy of a different state $|\psi\rangle$ in which every qubit is maximally entangled (as a singlet) with exactly one other qubit.

c. The monogamy property of quantum entanglement is clearly relevant for the study of quantum many-body systems. Can you give a (handwaving) argument why the monogamy property of entanglement gives rise to area laws for the entanglement entropy in ground states of quantum spin systems?

d. Prove that matrix product states (MPS) and projected entangled pair states (PEPS) with fixed bond dimensions χ satisfy an area law for the entanglement entropy.

3

a. Given a family of translational invariant nearest neighbour Hamiltonians $H = \sum_{i=-\infty}^{+\infty} h_{(i,i+1)}$ for 1-dimensional quantum spin chains. Assume that every term in the Hamiltonian is symmetric with respect to a symmetry G represented as a tensor product $\otimes_i U_i(g)$ for a faithful representation $U(g)$ of G ($\forall i, \left[h_{(i,i+1)}, U_i(g) \otimes U_{i+1}(g)\right] = 0$). How do you classify the different phases for the ground states of this family of Hamiltonians? What is the data that specifies those phases? What happens if you act with $\otimes_i U_i(g)$ on the ground states in those different phases?

b. Define a uniform injective matrix product state (MPS). Such states satisfy the fundamental theorem of MPS. Give a proof of this theorem.

c. Discuss the relevance of this theorem for the classification of phases of matter of quantum spins chains. In particular, consider the family of Hamiltonians exhibiting symmetries as in part a. of this question.

d. The AKLT state is an injective MPS defined on a spin chain of spin-1 systems for which the MPS tensors are given by the Pauli matrices $A^i = \sigma^i$. Specify a non-trivial global symmetry of that state under which the entanglement degrees transform according to a projective representation. What are the consequences for the entanglement spectrum?

4

a. Consider the Ising spin chain defined with periodic boundary conditions on a ring of L sites $H = \sum_{i=0}^{L} -X_i X_{i+1} + \lambda Z_i$ (here $X_{L+1} \equiv X_1$). What are the symmetries of this Hamiltonian? What is the ground state when $|\lambda| >> 1$? What about $|\lambda| << 1$? Argue why there must be a phase transition happening when interpolating between those extreme cases.

b. The 1-dimensional Kramers-Wannier duality applies to spin systems with the same symmetries as the Ising model. This duality transformation can be encoded in the form of an intertwiner that is of Matrix Product Operator (MPO) form. Write this MPO down in its graphical form, and define all the tensors involved. Argue that dual Hamiltonians have the same spectrum. What happens with the spectrum of the Ising Hamiltonian under this duality transformation?

c. Work out how the symmetries act on this MPO. What is the dual symmetry? What happens with the boundary conditions? What happens if you try to intertwine a non-symmetric operator?

d. The 2-dimensional Kramers-Wannier transformation has the form of a Projected Entangled Pair Operator and maps a quantum spin system defined on the vertices of a graph to a dual spin system defined on the edges of that graph. Make a graphical tensor network picture of this PEPO. The dual systems satisfies a 1-form symmetry; what is this symmetry, and why does it have to be there?

e. Argue that the toric code Hamiltonian is the Kramers-Wannier dual of the Ising model defined on the square lattice.

END OF PAPER