UNIVERSITY OF CAMBRIDGE

MAMA/342, NST3AS/342, MAAS/342, NST3PHY/2/TQM

MAT3

**MATHEMATICAL TRIPOS Part III**

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Monday 16 June 2025 9:00 am to 11:00 am

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# PAPER 342

# TOPOLOGICAL QUANTUM MATTER

## Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

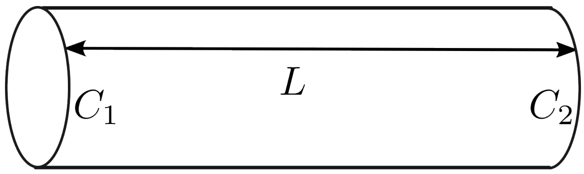
Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** This question considers topological order on the cylinder with boundaries $C_1$ and $C_2$, as shown below. The cylinder has length $L$ and circumference $C$.




Consider the surface code with Hamiltonian

$$H_0 = -J\sum_{v\in V} A_v - J\sum_{p\in P} B_p, \quad J>0, \quad A_v = \prod_{j\in v} X_j, \quad B_p = \prod_{j\in p} Z_j,$$

where qubits $j$ are placed on the links of a square lattice, $V$ is the set of vertices, $P$ is the set of plaquettes. We use a convention where an eigenvalue $a_v = -1$ for $A_v$ and $b_p = -1$ for $B_p$ signifies an $e$ anyon at $v$ and an $m$ anyon at $p$, respectively.

(a) Assuming that $C_1$ and $C_2$ are parallel with lattice links, define $A_v$ and $B_p$ at an $m$-condensing boundary. Explain your answer.

(b) Consider the perturbed surface code with Hamiltonian $H = H_0 + \delta H$ where $\delta H = h\sum_j X_j$ with $|h| \ll J$. Both $C_1$ and $C_2$ condense $m$. Obtain the ground state degeneracy for $h=0$. Obtain an estimate for the splitting of ground state energies as the function of $|h|/J$, $L$ and $C$. How (if at all) does the answer change if $C_1$ and $C_2$ condense $e$ instead?

Consider now matrix Chern Simons theory on the cylinder above, focusing on a theory with $4\times 4$ $K$ matrix

$$K = \begin{pmatrix} & M \\ M & \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Anyons are characterised by vectors $\mathbf{q} = (q_1, q_2, q_3, q_4)^{\mathrm{T}} \in \mathbb{Z}^4$. Encircling an anyon with $\mathbf{q}$ by an anyon with $\mathbf{q}'$ yields a phase factor $\exp(2i\theta_{\mathbf{q},\mathbf{q}'})$ with $\theta_{\mathbf{q},\mathbf{q}'} = \pi\mathbf{q}\cdot K^{-1}\mathbf{q}'$. Suppose that both $C_1$ and $C_2$ condense all anyons with $\mathbf{q}\in\mathcal{L}\subset\mathbb{Z}^4$, where $\mathcal{L}$ is the subspace of integer vectors with $q_3 = q_4 = 0$.

(c) Suppose that anyons with $\mathbf{q}$ condense at $C_1$ and consider the unitary operation $W_{\mathbf{q}}(x,x')$ of creating such an anyon at $x\in C_1$, dragging it into the bulk and returning it to $x'\in C_1$ to annihilate. For any ground state $|\psi\rangle$ of the system, $W_{\mathbf{q}}(x,x')|\psi\rangle = |\psi\rangle$ for any $x,x'\in C_1$ and any $\mathbf{q}\in\mathcal{L}$. By considering the commutator with $W_{\mathbf{q}'}(y,y')$ for suitable $y,y'\in C_1$ and anyon with $\mathbf{q}'$ (not necessarily in $\mathcal{L}$) that condenses at $C_1$, show that $\mathbf{q}' = \mathbf{u} + K\mathbf{l}$ where $\mathbf{u}\in\mathcal{L}$ and $\mathbf{l}\in\mathbb{Z}^4$. (The same holds for $C_2$.)

(d) Use the result from (c) to lower bound the ground state degeneracy of this system.

**2** This question is on quantum error correction and topological order, considering systems of $n$ qubits. Below we denote by $\mathcal{P}_n$ the corresponding $n$-qubit Pauli group, and consider a set $\{\tilde{E}_a | a = 1, \ldots, e\}$ of error operators.

(a) The Knill-Laflamme (KL) conditions can be formulated as

$$\Pi_{\mathcal{L}} \tilde{E}_a^{\dagger} \tilde{E}_b \Pi_{\mathcal{L}} = c_{ab} \Pi_{\mathcal{L}}, \quad \forall a, b \in \{1, \ldots, e\},$$

where $\Pi_{\mathcal{L}}$ is the projector onto the code space $\mathcal{L}$. Motivate the KL conditions by considering requirements for the set $\{\tilde{E}_a | a = 1, \ldots, e\}$ to be correctable.

Consider now quantum error correction with a stabilizer code of distance $d$.

(b) Define the centralizer $\mathcal{C}(\mathcal{S})$ of the stabilizer group $\mathcal{S}$. Express the set LOs of logical operators in terms of $\mathcal{C}(\mathcal{S})$ and $\mathcal{S}$. Show that for a set of error operators proportional to Pauli operators, $\tilde{E}_a \propto E_a \in \mathcal{P}_n$, the KL conditions hold if and only if

$$E_a^{\dagger} E_b \notin \text{LOs}, \quad \forall a, b \in \{1, \ldots, e\}.$$

(c) Show that from $\mathrm{wt}(E_a) < d/2$, $\forall a \in \{1, \ldots, e\}$ it follows that the KL conditions are satisfied. [Here $\mathrm{wt}(E_a)$ is the weight of $E_a$.] By providing a counterexample or otherwise, show that the converse does not hold.

Consider now the two-dimensional surface code with Hamiltonian

$$H_0 = -J \sum_{v \in V} A_v - J \sum_{p \in P} B_p, \quad J > 0, \quad A_v = \prod_{j \in v} X_j, \quad B_p = \prod_{j \in p} Z_j,$$

where qubits $j$ are placed on the links of a square lattice, $V$ is the set of vertices, $P$ is the set of plaquettes. Suppose that the system is on a manifold such that it has a ground space furnishing $k$ logical qubits.

(d) Show that the projector $\Pi_{\mathcal{L}}$ to the ground space $\mathcal{L}$ of $H_0$ satisfies the KL conditions for any "suitably local" error set with $\tilde{E}_a \propto E_a \in \mathcal{P}_n$. Define what "suitably local" means in terms of the code distance $d$ and the supports of $E_a$.

(e) Consider now the perturbed surface code with Hamiltonian $H = H_0 + \delta H$ where $\delta H = h_X \sum_j X_j + h_Y \sum_j Y_j + h_Z \sum_j Z_j$ with $0 < |h_{X,Y,Z}| \ll J$. By invoking local unitary equivalence, show that there exist $n - k$ independent stabilizer generators $\tilde{S}_j$ such that $\tilde{S}_j |\varphi_\alpha\rangle = |\varphi_\alpha\rangle$ for the $2^k$ lowest-energy eigenstates $|\varphi_\alpha\rangle$ of $H$, but now $\tilde{S}_j$ are only quasilocal and $\tilde{S}_j \notin \mathcal{P}_n$. Show that for any set of Pauli errors, the subspace spanned by these $|\varphi_\alpha\rangle$ can satisfy the KL conditions at most approximately.

**3** This question is on fermions, Majorana zero modes (MZMs), and their use for quantum computing.

(a) Define fermion parity, fermion-parity-even operators, and fermion-parity-odd operators. By considering the compatibility of measurements in space-like separated regions or otherwise, show that local observables must be fermion-parity even.

(b) Consider the mean-field description of superconductors in terms of the Hamiltonian

$$H = \frac{1}{2}\left(\mathbf{a}^\dagger, \mathbf{a}\right) H_{\mathrm{BdG}} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}, \quad H_{\mathrm{BdG}} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^* \end{pmatrix}.$$

Suppose that $H_{\mathrm{BdG}}$ has locally nondegenerate zero-energy solutions $\psi_j$, exponentially localised around $\mathbf{x}_j$ with decay length $\xi$, and separated by distance $\ell = \min_{i\neq j} |\mathbf{x}_i - \mathbf{x}_j|$. Working in the $\ell/\xi \to \infty$ limit, show that the $\psi_j$ give rise to operators $\gamma_j$ (MZMs) that, when suitably normalised, satisfy

$$\gamma_j = \gamma_j^\dagger, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}.$$

Show that a system with $2M$ such MZMs furnishes a groundspace $\mathcal{L}$ with $k = M-1$ topological qubits.

(c) Show that for any local observable $\mathcal{O}$ and orthonormal basis $\{|\varphi_\alpha\rangle\}$ in $\mathcal{L}$, we have $\langle\varphi_\alpha|\mathcal{O}|\varphi_\beta\rangle = c_{\mathcal{O}}\delta_{\alpha\beta}$ with constant $c_{\mathcal{O}}$. Explain how and why the result changes for $1 \ll \ell/\xi < \infty$.

The rest of the question considers quantum computing with these $k = M-1$ topological qubits for $M \geqslant 3$ (i.e., we use the dense encoding).

(d) Suppose that MZMs $\gamma_{2M}$ and $\gamma_{2M-1}$ are used as ancillas such that the system is in a state $|\psi\rangle$ satisfying $i\gamma_{2M}\gamma_{2M-1}|\psi\rangle = |\psi\rangle$. Show that

$$\exp(i\pi\gamma_1\gamma_2\gamma_3\gamma_4/4)|\psi\rangle \propto \exp(\pi\gamma_3\gamma_{2M}/4)\frac{\mathbb{1} - i\gamma_3\gamma_{2M-1}}{2}\frac{\mathbb{1} + \gamma_1\gamma_2\gamma_4\gamma_{2M-1}}{2}|\psi\rangle.$$

(e) Show that $\Upsilon_{123} = i\gamma_1\gamma_2\gamma_3$ and $\gamma_4$ satisfy the same algebraic relations as a pair of Majorana fermions. Hence show that $\exp(\pi\Upsilon_{123}\gamma_4/4)$ can be used to exchange $\Upsilon_{123}$ and $\gamma_4$. Using this and the result from (d), show that using braids $R_{ab} = \exp(\pi\gamma_a\gamma_b/4)$ and suitable fermion-parity measurements one can map, in the sense of the action on $|\psi\rangle$, the MZM $\gamma_1$ to any Hermitian fermion-parity-odd MZM monomial of $\gamma_j$ ($j \in \{1, \ldots 2M-2\}$).

## END OF PAPER