UNIVERSITY OF CAMBRIDGE

MAMA/337, NST3AS/337, MAAS/337

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 17 June 2025 1:30 pm to 3:30 pm

PAPER 337

APPLICATIONS OF QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The different parts of this question are unrelated.

(a) Consider a superfluid in three spatial dimensions in which a $U(1)$ symmetry is spontaneously broken. Write down the Landau-Ginzburg effective Lagrangian for the complex order parameter ψ and show that the number density is conjugate to the phase. What does this imply for states with spontaneous symmetry breaking?

(b) Let $|\uparrow\rangle$ be the highest weight state of the spin S irreducible representation of $SU(2)$. Define $|g\rangle \equiv g|\uparrow\rangle$, where g is an element of $SU(2)$ in this representation. Show that

$$\int dg\, |g\rangle\langle g| = c\, \mathrm{Id}\,,$$

where Id is the unit operator in the representation and c is a constant that you do not need to determine. Use this result to obtain the coherent state path integral representation for $\langle g(t)|g(0)\rangle$ for a single non-interacting spin S degree of freedom, involving the Wess-Zumino term $\langle \frac{dg}{dt}|g\rangle$.

By parametrising

$$|g\rangle = e^{-i\phi S_3} e^{-i\theta S_2} e^{-i\psi S_3}|\uparrow\rangle\,,$$

obtain an explicit expression for the Wess-Zumino term in terms of the angles ϕ, θ, ψ.

Show that the Wess-Zumino term in the path integral is independent of ψ and derive, from the path integral, why the coefficient S in this term must be quantised.

(c) Recall that the retarded Green's function of an operator A at temperature T and frequency ω is given by

$$G^R_{AA}(\omega) = -i\int_0^\infty dt\, e^{i\omega t}\mathrm{Tr}\left(e^{-H/T}[A(t), A(0)]\right)\,.$$

Here H is the Hamiltonian of the system. There is no spatial dependence in this question. Obtain an expression for $G^R_{AA}(\omega)$ in terms of a double sum over the eigenvalues E_n of the Hamiltonian H and the overlaps $\langle m|A(0)|n\rangle$ of the operator between the corresponding eigenstates of H.

Consider the case of a simple harmonic oscillator, where $H = \varepsilon a^\dagger a$, with ε a constant. The raising operator acts as $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Obtain the retarded Green's function for $A = a^\dagger$ and interpret your result physically.

2

This question is concerned with a system in three spatial dimensions with a Fermi surface.

(a) The low energy effective action for a spinless Fermi liquid is

$$S = \int dt d^2k d\ell \left[i\psi^\dagger(p) \frac{d\psi(p)}{dt} - \ell v_F(k) \psi^\dagger(p)\psi(p) \right] .$$

Here k parametrise directions along the Fermi surface and ℓ is perpendicular to the Fermi surface. We have $p = k + \ell$. The Fermi velocity is $v_F(k)$. Describe a scaling towards the Fermi surface under which this term is marginal. Consider a quadratic deformation, parametrised by the function $\mu(k)$,

$$\delta S = \int dt d^2k d\ell \, \mu(k) \psi^\dagger(p)\psi(p) .$$

Is this deformation marginal, irrelevant or relevant? What is the effect of this term?

Now consider a general *quartic* interaction and explain why it is irrelevant under this scaling. What is the condition on the momenta of the particles for a quartic interaction to be marginal? You do not need to discuss these cases in detail.

(b) Compute the one-loop running of the BCS coupling. That is, given the interaction

$$\delta S = V \int dt d^2k d^2k' d\ell d\ell' \, \delta(\ell + \ell') \psi^\dagger(p) \psi^\dagger(-p) \psi(p') \psi(-p') ,$$

with V a constant, consider the scattering of two fermions with momenta p and $-p$, both with (imaginary time) frequency ω. Use the imaginary time propagator

$$\langle \psi^\dagger \psi \rangle (\omega, p) = \frac{1}{i\omega - v_F(k)\ell} ,$$

and assume that the Fermi surface has a symmetry such that $v_F(-k) = -v_F(k)$. Show that the tree level answer for the scattering is V and that the one-loop correction exhibits a logarithmic divergence in ω. By resumming the logarithms, obtain an expression for the frequency scale ω at which an attractive coupling ($V < 0$) becomes strong. What will happen at that scale?

(c) The zero temperature BCS gap equation is given by

$$\frac{1}{|V|} = \int \frac{d^2k d\ell d\omega}{(2\pi)^4} \frac{1}{\omega^2 + v_F(k)^2 \ell^2 + \Delta^2} .$$

Here V is the same constant as above. Obtain a formula for the gap Δ in the limit in which Δ is much smaller than any high energy cutoff scale that you need to introduce. How does your result relate to the strong coupling scale you obtained in the previous part of this question?

END OF PAPER