MAMA/336, NST3AS/336, MAAS/336

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 9:00 am to 11:00 am

PAPER 336

PERTURBATION METHODS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider

$$I(a) = \int_0^\infty \frac{dx}{(a+x)^{1/2}(1+x)}.$$

Use the divide and conquer method to show that, for $a \to 0$,

$$I(a) \sim \pi - 2\sqrt{a} + \pi a/2.$$

You may note that $\arctan x \sim x - x^3/3$ as $x \to 0$.

(b) The function $f(\lambda)$ is defined as

$$f(\lambda) = \int_{\mathcal{C}} e^{\lambda((z-1)^3+3(z-1))} dz$$

where λ is a large positive real constant and $\mathcal{C}$ is a contour extending from $-\infty$ to $+\infty e^{i\psi}$ in the sector where

$$\frac{\pi}{6} < \psi < \frac{\pi}{2}.$$

Locate the saddle points $z_\pm$ for the exponent function $\phi(z) = (z-1)^3 + 3(z-1)$ in the complex plane, and also sketch lines of constant $\Re(\phi(z))$, including those passing through the saddles (i.e. the stationary phase lines), and the contours of steepest descent/ascent given by $\Im(\phi(z)) = \Im(z_\pm)$. Label the 'hills' and 'valleys'.

By deforming $\mathcal{C}$ onto the appropriate steepest descent path, show that

$$f(\lambda) \sim \sqrt{\frac{\pi}{3\lambda}} e^{i\pi/4+2i\lambda}.$$

Now suppose that $\mathcal{C}$ starts from $z = 1$ and tends to ∞ in the same sector as above. Identify the appropriate path that should be taken to obtain asymptotic contributions from the saddle and the end point, and hence deduce that

$$f(\lambda) \sim \sqrt{\frac{\pi}{3\lambda}} e^{i\pi/4+2i\lambda} - \frac{1}{3\lambda}.$$

2 (a) A system variable, $u(t)$ is governed by the equation

$$\frac{d^2u}{dt^2} + (1 + \epsilon^2 f + \epsilon \sin t)u = 0,$$

where f is a real-valued constant and $0 < \epsilon \ll 1$. Use the method of multiple scales to find the range of values of f for which solutions to this equation remain bounded (at least up to order unity in the slow time $T = \epsilon^2 t$), irrespective of the initial conditions.

(b) The flow of heat through an inhomogeneous bar is governed by the diffusion equation, which, in non-dimensionalised form is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{d(x/\epsilon)}\frac{\partial u}{\partial x}\right),$$

where u is the temperature and $d(X)$ is a non-negative periodically-varying coefficient, with period 1, that is related to the diffusion. Note that $d(X)$ may be a discontinuous function, but the temperature and heat flux are continuous, and $d(X)$ has (finite) mean

$$\bar{d} = \int_0^1 d(X)dX.$$

Derive the homogenized, or mean-field, equation when $\epsilon \to 0$.

Suppose $d(X)$ is a sawtooth function (i.e. a piecewise-linear periodic function) with profile

$$d(X) = \begin{cases} 2X/p, & 0 \leqslant X \leqslant p, \\ 2(1-X)/(1-p), & p \leqslant X \leqslant 1. \end{cases}$$

Determine $\bar{d}$, and show by direct substitution or otherwise that the solution of the homogenized bar with conditions

$$u(x,0) = 0; \qquad u(x,t) \to 0 \text{ for } x \to \infty; \qquad u(0,t) = T_0 \text{ for } t > 0,$$

takes the similarity form

$$u(x,t) = T_0\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right), \qquad \text{where } \operatorname{erf}(\zeta) = \frac{2}{\sqrt{\pi}}\int_0^\zeta e^{-z^2}dz.$$

3 (a) Derive the leading order WKB solution of the equation

$$\frac{d^2y}{dx^2} + k^2(\epsilon x)y = 0,$$

for $\epsilon \to 0$, and specify restrictions on its validity. Use this to determine the asymptotic solution in $x > 0$ for the case $k^2(X) = 1 + X^2$, $y(0) = 1$, $\frac{dy}{dx}(0) = 0$.

In the last part you may find the substitution $X = \sinh u$ useful.

(b) Consider the boundary value problem

$$\epsilon\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + xy = 0, \qquad y(0) = 0, \quad y(1) = 2.$$

If $\epsilon \to 0$ identify the location of an inner layer.

The first two terms of the outer solution are, in the usual notation, given as:

$$y^{(1)}(x) = y_0(x) + \epsilon\, y_1(x) = (1+x)e^{1-x} + \epsilon\left[-(1+x)\ln(1+x) + (1+x)\ln 2 + (x-1)\right]e^{1-x}.$$

By matching, find the first two non-zero terms in the inner expansion. Hence determine the uniformly-valid additive composite expansion, which is correct to $O(\epsilon)$.

END OF PAPER